

Optimal design problems in rough inhomogeneous media. Existence theory^{*}

Eduardo V. Teixeira

Rutgers University,
Department of Mathematics,
Piscataway, NJ 08854-8019

Abstract

This paper settles the existence question for a rather general class of convex optimal design problems with a volume constraint. In low dimensions, we prove the existence of an optimal configuration for general convex minimization problems ruled by bounded measurable degenerate elliptic operators. Under a mild continuity assumption on the medium, the free boundary is proven to enjoy the appropriate weak geometry and we establish the existence of an optimal design for general convex optimal design problems with volume constraints for all dimensions.

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1 Introduction

Well known for modeling important problems in applied mathematics, respected for the challenging mathematical questions they give rise to and admired for their intrinsic beauty, optimization problems with volume constraints have received an overwhelming attention in the past few decades. In general, the usual techniques of the Calculus of Variations are not sufficiently powerful, or even appropriate, to establish existence of optimal configurations for those classes

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of problems. This fact has inspired remarkable recent advances in a number of branches of applied analysis in an attempt to develop the right set of analytical and geometrical tools to study optimal design problems with volume constraints.

One of the fundamental motivations of this present work can be, in its most basic form, stated as follows: given an n -dimensional body and a fixed amount of insulating material, what is the best way of insulating it? Depending on the flexibility allowed, the mathematical set-up used to model this classic question can also be employed in the analysis of a variety of other problems in applied mathematics. In more precise mathematical terms, but still using the language of heat conduction, the above question takes the following form: let D be a fixed Lipschitz bounded domain in \mathbb{R}^n (the body to be insulated), $\varphi: \partial D \rightarrow \mathbb{R}$ be a prescribed positive function (the temperature distribution on D), and $\iota > 0$ be a given positive number (the amount of insulating material available). For each configuration Ω that surrounds D and obeys $\mathcal{L}^n(\Omega \setminus D) \leq \iota$, we compute the flux associated to it:

$$\Omega \mapsto \mathfrak{J}(\Omega).$$

In general, \mathfrak{J} is related to a boundary integral involving a potential u_Ω , linked to Ω by a prescribed PDE. The optimal design problem is then

$$\text{Min } \{ \mathfrak{J}(\Omega) \mid \Omega \subset D \text{ and } \mathcal{L}^n(\Omega \setminus D) \leq \iota \}. \quad (1.1)$$

Probably the first and still one of the most influential works in this line of research is the pioneering article of Aguilera, Alt and Caffarelli, [AAC86]. In this paper, the authors address the question of minimizing the Dirichlet integral when prescribed the volume of the zero set. More precisely, they study the optimization problem

$$\text{Min } \left\{ \int |\nabla u|^2 dX \mid u \in H^1(\Omega), \quad u = \varphi \geq 0 \text{ on } \partial\Omega \quad \text{and} \quad \mathcal{L}^n(\{u = 0\}) = \alpha \right\}, \quad (1.2)$$

for a fixed $\alpha < \mathcal{L}^n(\Omega)$. In the case of an exterior domain, $\Omega = \mathbb{R}^n \setminus D$, problem (1.2) can be used to model a very simple, yet interesting optimal design problem with volume constraint as stated above. Namely, suppose D is evenly heated. If one tries to minimize the heat flux given by $\int_{\partial\Omega} u_\mu d\mathcal{H}^{n-1}(X)$, where u is the capacity potential associated to Ω , with $\mathcal{L}^n(\Omega \setminus D)$ prescribed, a simple application of Green's identity reveals that the heat flux equals the Dirichlet integral, and therefore the problem becomes identical to (1.2). Fine regularity properties of the free boundary, $\partial\{u^* > 0\} \cap \Omega$, where u^* is a minimizer of (1.2) rely on the powerful geometric-measure machinery developed by Alt and Caffarelli in [AC81]: the *magnum opus* of free boundary regularity theory for variational problems.

A significant generalization of problem (1.2) was carried out by Lederman in [Led96]. In this paper, the author studies the non-homogeneous minimization problem, that is, the Dirichlet integral is replaced by $\int |\nabla u|^2 dX - \int g u$, for a given g bounded away from zero.

In an important paper, Ambrosio, Fonseca, Marcellini and Tartar, [AFMT99], address another major generalization of problem (1.2). Namely they establish the existence of a minimizer to the functional $\mathcal{F} := \int_\Omega W(Du) dx$, for $W: \mathbb{R}^{d \times n} \rightarrow (0, \infty)$ C^1 and quasi-convex, with the multiple volume constraint $\mathcal{L}^n(\{u = z_i\}) = \alpha_i, 0 \leq i \leq k$. In a subsequence article, Tilli, in [Tilli99], showed, for $W(\xi) := |\xi|^2$, that in the case of just two level constraints, the minimizers are locally Lipschitz continuous.

Still assuming a constant temperature distribution, Oliveira and the author in [OT06] studied the optimization problem (1.1), governed by the p -Laplacian operator when the flux is given by $\mathfrak{J}(u) := \int_{\partial\Omega} (u_\mu)^{p-1} d\mathcal{H}^{n-1}(X)$. This translates into the analysis of the minimization problem (1.2), for the p -Dirichlet integral, that is, $W(\xi) = |\xi|^p$, for $p > 1$.

The first work to deal with optimal design problems with non-constant temperature distribution $\varphi: \partial D \rightarrow (0, \infty)$ is [ACS87]. In this paper, the authors consider the linear functional: $\mathfrak{J}(\Omega) = \int \Delta u dX$, where u is the harmonic function in $\Omega \setminus D$, taking boundary data φ on ∂D and zero on $\partial\Omega$. Even for this simple functional, major difficulties arise. For instance, the free

boundary condition, that is, the behavior of ∇u^* along the free boundary, $\partial\Omega^*$, is non-local and it required a new machinery to establish the appropriate geometric-measure properties of the free boundary necessary to perform suitable smooth perturbations. The latter is used in its entirety to finally conclude the existence of an optimal design.

At least for smooth competing configurations, Ω , for the linear functional studied in [ACS87] we have

$$\mathfrak{J}(\Omega) := \int \Delta u dX = \int_{\partial\Omega} u_\nu d\mathcal{H}^{n-1}(X) = \int_{\partial D} u_\mu d\mathcal{H}^{n-1}(X).$$

This is a naïve, yet important observation, as the latter integral is taken over the fixed boundary. Therefore, at least in an intuitive perspective, a non-linear theory for this class of minimization problems should use $\int_{\partial D} u_\mu d\mathcal{H}^{n-1}(X)$ as its linear pattern. From the applied viewpoint, if one allows a nonlinear flux, \mathfrak{J} that might also depend upon the local structure of the boundary of the body D , i.e.,

$$\mathfrak{J}(\Omega) := \int_{\partial D} \Gamma(X, u_\mu(X)) d\mathcal{H}^{n-1}(X) \quad (1.3)$$

the mathematical model (1.1) would address several other physical situations, such as: optimal configurations in electrostatics, problems in material science, flux dynamics, among many others. This nonlinear setting, however still only for problems governed by the Laplacian operator, has been studied by the author in [Teix05] and [Teix07].

In this present paper, we settle the existence theory for optimal design problem (1.1) with nonlinear functionals as in (1.3), when u_Ω is linked with Ω by a rather general class of degenerate elliptic PDEs. In terms of applications, it greatly extends the range of physical systems that can be modeled by this set-up. From the mathematical viewpoint, this project brings a number of new rather challenging difficulties in its analysis and modern solutions to various issues commonly found in free boundary problems are developed throughout the paper. Free boundary regularity theory for uniform elliptic operators in divergence form with merely Hölder continuous coefficients is currently being developed in order to establish $C^{1,\gamma}$ smoothness of an optimal configuration, up to a possible negligible singular set, [Teix-Prep].

The article is organized as follows: in section 2, we describe all the mathematical elements involved in the model and the optimization problem is accurately stated in that section. Still in section 2, we introduce weak formulations of the optimal design problem (1.1) that are somewhat simpler to be tackled from the mathematical perspective. Basic properties of the functional to be minimized are established in section 3. The first existence theorem for a weak formulation of the original optimization problem is delivered in section 4. In section 5, by letting the penalty term blow-up, we establish the existence of an optimal configuration to the optimal design problem with volume constraint (1.1) ruled by totally discontinuous degenerate elliptic operators. For that though, a technical restriction on the dimension is necessary. In section 6, under C^ϵ regularity on the medium, a series of results concerning the weak geometric properties of the boundary of an optimal configuration to the weak formulation of the original problem (1.1) are achieved. These are used in section 7 to ultimately derive existence of an optimal configuration in all dimensions.

2 Mathematical set-up

Throughout the paper, D denotes a fixed Lipschitz bounded domain in \mathbb{R}^n , $\varphi: \partial D \rightarrow \mathbb{R}$ is a prescribed positive function and $\iota > 0$ is a given positive number. Our medium deformation will be expressed by $\mathcal{A}: D^C \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, a measurable p -degenerate elliptic map, that is,

- (a) for each $\xi \in \mathbb{R}^n$, the mapping $X \mapsto \mathcal{A}(X, \xi)$ is measurable.
- (b) For a.e. $X \in D^C$, the mapping $\xi \mapsto \mathcal{A}(X, \xi)$ is continuous.
- (c) There exists constants $0 < \lambda \leq \Lambda < \infty$ and a $p > 1$, such that, for a.e. $X \in D^C$ and all $\xi \in \mathbb{R}^n$,

- (i) $\mathcal{A}(X, \xi) \cdot \xi \geq \lambda |\xi|^p$,
- (ii) $|\mathcal{A}(X, \xi)| \leq \Lambda |\xi|^{p-1}$,
- (iii) $\langle \mathcal{A}(X, \xi_1) - \mathcal{A}(X, \xi_2), \xi_1 - \xi_2 \rangle > 0$, whenever $\xi_1 \neq \xi_2$ and
- (iv) $\mathcal{A}(X, \alpha \xi) = \alpha |\alpha|^{p-2} \mathcal{A}(X, \xi)$.

A typical example to keep in mind is

$$\mathcal{A}(X, \xi) = A(X) |\xi|^{p-2} \xi,$$

with A bounded measurable, which gives rise to the theory of optimal shape problems governed by the p -Laplacian in a totally discontinuous medium.

Our optimization problem is then formulated as follows: for each domain $\Omega \subset D$ satisfying

$$\mathcal{L}^n(\Omega \setminus D) \leq \iota, \quad (2.1)$$

we consider the \mathcal{A} -potential, $u = u(\Omega)$, with the prescribed boundary value φ on the fixed boundary ∂D , associated to Ω , i.e. the unique solution to

$$\begin{cases} \mathbb{L}u := \operatorname{div}(\mathcal{A}(X, Du)) &= 0 \text{ in } \Omega \setminus D \\ u &= \varphi \text{ on } \partial D \\ u &= 0 \text{ on } \partial \Omega \end{cases} \quad (2.2)$$

and compute

$$\mathfrak{J}(\Omega) := \int_{\partial D} \Gamma(X, \partial_{\mathcal{A}} u(X)) d\mathcal{H}^{n-1}(X) \quad (\text{the flux: quantity to be minimized}).$$

Here $\Gamma: \partial D \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function, whose properties will be described soon, and

$$\partial_{\mathcal{A}} u(X) := \langle \mathcal{A}(X, \nabla u(X)), \mu(X) \rangle \quad (2.3)$$

where μ denotes the inward normal vector defined \mathcal{H}^{n-1} a.e. on ∂D . The optimal design problem we are interested in is the following:

$$\text{Minimize } \left\{ \mathfrak{J}(\Omega) \mid \Omega \supset D \text{ and } \mathcal{L}^n(\Omega \setminus D) \leq \iota \right\}. \quad (2.4)$$

The analytical (and naturally mild) properties assumed on the nonlinearity Γ are:

1. For each $X \in \partial D$ fixed, $\Gamma(X, \cdot)$ is convex and increasing.
2. For each $t \in \mathbb{R}$ fixed, $\partial_t \Gamma(\cdot, t)$ is continuous.
3. If $\Gamma(X_0, t_0) = 0$ then $\Gamma(Y, t_0) = 0 \forall Y \in \partial D$; otherwise $\frac{\Gamma(Y, t)}{\Gamma(X, t)} \leq L$, for a universal constant $L > 0$.

Notice that from 1 the following coercivity condition holds:

$$\lim_{t \rightarrow +\infty} \int_{\partial D} \Gamma(X, t) d\mathcal{H}^{n-1}(X) = +\infty. \quad (2.5)$$

If ψ is a positive continuous function defined on ∂D and γ is a increasing convex function, then

$$\Gamma(X, t) = \psi(X) \gamma(t)$$

gives a typical nonlinearity that fulfils the above properties. As in the Calculus of Variations, Γ is chosen based upon the particular problem we are trying to model and no relation whatsoever is imposed upon the nonlinearity Γ and \mathcal{A} .

Sometimes it is convenient to use the language of heat conduction theory to describe the elements involved in our analysis. Thus, D is the body to be insulated, φ represents the temperature distribution on ∂D , ι corresponds to the maximum amount of insulating material available, \mathfrak{J} plays the role of the (generalized) heat flux, which is the quantity to be minimized, and \mathcal{A} determines the inhomogeneous and complexity features of the medium. However it is important to highlight that this model is widely applicable to several other situations beyond the bounds of the classical heat conduction theory and other interpretation of the model might provide different insights on what is reasonable to expect to hold.

It is noteworthy to point out that, since we are not forcing any regularity assumption on the medium \mathcal{A} , in principle just Hölder continuity estimates are available for an \mathcal{A} -potential $u = u(\Omega)$. Thus, the \mathcal{A} -normal derivative of u , $\partial_{\mathcal{A}} u$, as entitled in (2.3) is not properly defined. Some of our primary results concerning geometric properties of the free boundary will not depend upon any smoothness condition on the medium. However, just to grapple with this technical inconsistency, we will assume throughout the paper that there exists a small $1 \gg \delta_0 > 0$, such that

- (i) \mathcal{A} is Hölder (or even only Dini) continuous in $D_{\delta_0} := \{X \in \mathbb{R}^n \mid \text{dist}(X, \partial D) < \delta_0\}$ and
 - (ii) $\varphi: \partial D \rightarrow \mathbb{R}$ is accordantly smooth.
- (2.6)

Once more we emphasize that for the first part of this project, condition (2.6) plays merely a technical role and, for sake of applications, it should not be seen as a constraint.

2.1 Penalty Method and weak formulation

From the mathematical point of view, the minimization problem (2.4) carries too many difficulties to be approached directly. Instead, we will employ a fruitful penalty method in order to formulate weak versions of problem (2.4). Such a technique has been successfully employed to study a variety problems in applied mathematics.

The intuitive idea behind a penalization strategy is the following: suppose our problem has an “undesired” (from the mathematical perspective) constraint on the competing configurations (in our case a volume constraint). We then allow any configuration to compete; however we “charge a fee” for those configuration that do not obey the previously set constraint. We expect that, if the fee is too high, optimal configurations will indeed prefer to satisfy the original constraint.

Still in a philosophical perspective, one should expect that an optimal configuration, Ω^* , of problem (2.4) satisfies

$$\mathcal{L}^n(\Omega^* \setminus D) = \iota.$$

For that, think of ι as the budget available and Ω^* as the ultimate object to be built up. Mathematically, this fact is indeed easily justified. For instance suppose, for an optimal configuration Ω^* , we had

$$\mathcal{L}^n(\Omega^* \setminus D) < \iota - \varepsilon,$$

for some $\varepsilon > 0$. Let $X_0 \in \partial\Omega^*$ be a free boundary point and $\rho > 0$ so that $\omega_n \rho^n < \varepsilon$. Consider

$$\tilde{\Omega} := \Omega^* \cup B_\rho(X_0).$$

Thus, $\tilde{\Omega}$ competes with Ω^* in the minimization problem (2.4) and, because of maximum principle, $u(\tilde{\Omega}) > u(\Omega^*)$. Taking into account that Γ is increasing and applying Hopf maximum principle on ∂D , we would conclude

$$\mathfrak{J}(\Omega^*) > \mathfrak{J}(\tilde{\Omega}),$$

which contradicts the minimality property of Ω^* . Our conclusion is that in problem (2.4) we can regard the condition $\mathcal{L}^n(\Omega \setminus D) \leq \iota$ as $\mathcal{L}^n(\Omega \setminus D) = \iota$. For future reference, let us state this as a Lemma.

Lemma 2.1. *Let Ω^* be a minimizer of problem (2.4). Then $\mathcal{L}^n(\Omega^* \setminus D) = \iota$.*

Another general comment: we will always extend the \mathcal{A} -potential $u(\Omega)$ by zero outside Ω . Thus, in the distributional sense,

$$\mathbb{L}[u(\Omega)] = 0, \text{ in } \Omega = \{u(\Omega) > 0\} \quad \text{and} \quad \mathbb{L}[u(\Omega)] \geq 0, \text{ in } \mathbb{R}^n \setminus D. \quad (2.7)$$

Returning to the penalty technique issue: we shall borrow the simple, yet quite clever penalty term suggested in [Tilli99], that is, for each $\lambda > 0$, we will consider the penalization term $\varrho_\lambda: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, defined by

$$\varrho_\lambda(t) := \lambda(t - \iota)^+. \quad (2.8)$$

We then define the λ -perturbed functional, \mathfrak{J}_λ , to be

$$\mathfrak{J}_\lambda(\Omega) := \int_{\partial D} \Gamma(X, \partial_{\mathcal{A}} u(X)) d\mathcal{H}^{n-1}(X) + \varrho_\lambda(\mathcal{L}^n(\Omega \setminus D)). \quad (2.9)$$

Once more, the idea is the following: we allow \mathfrak{J}_λ to act on any configuration $\Omega \supset D$ and, when λ is big enough, we hope that an optimal design Ω_λ^* for \mathfrak{J}_λ will satisfy $\mathcal{L}^n(\Omega_\lambda^* \setminus D) = \iota$, thus it will also be a minimizer for our original optimization problem with volume constraint. Our initial goal is then study existence and geometric properties of the penalized problem:

$$(\mathfrak{P}_\lambda) \quad \text{Minimize } \left\{ \mathfrak{J}_\lambda(\Omega) \text{ among all sets } \Omega \supset D \right\}. \quad (2.10)$$

However, even the penalty problem (2.10) is, in principle, too hard to be directly approached. Thus, for the time being, it will be more appropriate to initially deal with a weak formulation of problem (2.10), which we start describing now. Let δ_0 be the technical number in (2.6). For each $\delta \ll \delta_0$, we us define the functional set

$$\mathcal{V}(\delta) := \{f \in W^{1,p}(D^C) \mid f = \varphi \text{ on } \partial D, f \geq 0, \mathcal{L}f \geq 0, \mathcal{L}f = 0 \text{ in } D_\delta\}. \quad (2.11)$$

Then we define the sample functional set:

$$\mathcal{V} := \bigcup_{\delta \searrow 0} \mathcal{V}(\delta) \quad (2.12)$$

and the weak formulation of problem (2.10) can then be stated as

$$(\mathfrak{P}_\lambda^{\text{weak}}) \quad \min_{f \in \mathcal{V}} \left\{ \int_{\partial D} \Gamma(X, \partial_{\mathcal{A}} f(X)) d\mathcal{H}^{n-1}(X) + \varrho_\lambda(\mathcal{L}^n(\{f > 0\} \setminus D)) \right\}. \quad (2.13)$$

3 Basic functional and analytic properties

In this section we establish all the basic and necessary properties on the mathematical elements of the problems we are interested in, namely, problems (2.4), (2.10) and (2.13); however we will mostly be concerned with the latter, as it the the weakest formulation among them all.

We start by stating, as a lemma, a simple yet crucial observation regarding the measure theory involved on our optimization problems. The proof is somewhat long, but rather standard and we omit it here.

Lemma 3.1. *Let $f \in \mathcal{V}$, as in (2.12). Then $\mathcal{L}f$ defines a nonnegative Radon measure, μ_f , in D^C . In particular, for any $\psi \in C(D^C) \cap W^{1,p}(D^C)$,*

$$\int_{B_r(Y)} \psi(X) d\mu_f(X) + \int_{B_r(Y)} \langle \mathcal{A}(X, Df), D\psi(X) \rangle dX = \int_{\partial B_r(Y)} \psi(S) \cdot \partial_{\mathcal{A}} f(S) d\mathcal{H}^{n-1}(S),$$

for almost all $0 \leq r < \text{dist}(Y, \partial D)$. Also, if $\psi \in W_0^{1,p}(D^C)$, there holds

$$\int_{D^C} \langle \mathcal{A}(X, Df), D\psi(X) \rangle dX = \int_{D^C} \psi(X) d\mu_f(X). \quad (3.1)$$

Furthermore,

$$\mu_f(\mathbb{R}^n \setminus D) = \int_{\partial D} \partial_{\mathcal{A}} f(X) d\mathcal{H}^{n-1}(X). \quad (3.2)$$

Another useful results that we make use throughout the paper is (for a proof in the case of the p -Laplacian we refer, for instance, to [DP05], page 100):

Lemma 3.2. *Let \mathcal{O} be a domain in \mathbb{R}^n and $f \in W^{1,p}(\mathcal{O})$. There exists a constant $c = c(n, \mathcal{A}) > 0$, such that*

$$\int_{\mathcal{O}} (\langle \mathcal{A}(X, Df), Df \rangle - \langle \mathcal{A}(X, Dh), Dh \rangle) dX \geq c \begin{cases} \int_{\mathcal{O}} |\nabla(f-h)|^p dX & \text{if } p \geq 2 \\ \alpha(f) \cdot \left[\int_{\mathcal{O}} |\nabla(f-h)|^p dX \right]^{2/p} & \text{if } 1 < p \leq 2. \end{cases}$$

where

$$\alpha(f) := \left[\int_{\mathcal{O}} |\nabla f|^p dX \right]^{1-\frac{2}{p}}$$

and h is the \mathcal{A} -harmonic function in \mathcal{O} that agrees with f on $\partial \mathcal{O}$.

Our first Proposition provides an energy estimate for a minimizing sequences to our optimization problems. More precisely, we have:

Proposition 3.3. *Let u_j be a minimizing sequence for the functional \mathfrak{J}_λ . Then,*

$$\|\nabla u_j\|_{L^p(D^C)} \leq C,$$

where C depends only on dimension, \mathcal{A} , D , φ and Γ .

Proof. Let $\mathfrak{h} = \mathfrak{h}_p$ be the p -harmonic function in D^C that agrees with φ on ∂D^C , that is the solution to

$$\begin{cases} \Delta_p \mathfrak{h} &= 0 \text{ in } D^C \\ \mathfrak{h} &= \varphi \text{ on } \partial D^C \\ \mathfrak{h} &\in W^{1,p}(D^C). \end{cases} \quad (3.3)$$

From the maximum principle, there holds

$$0 \leq \mathfrak{h} \leq \sup_{\partial D} \varphi.$$

For sake of notation convenience, let us denote $\int \mathfrak{h} d\mu_j := \mu_{u_j} = \mu_j$, as in Lemma 3.1. We clearly have

$$\begin{aligned} \int (\mathfrak{h} - u_j) d\mu_j &= \int \langle \mathcal{A}(X, Du_j), (\mathfrak{h} - u_j) \rangle dX \\ &= \int \langle \mathcal{A}(X, Du), Dh(X) \rangle dX - \int \langle \mathcal{A}(X, Du), Du \rangle dX. \end{aligned} \quad (3.4)$$

From the degenerate ellipticity of \mathcal{A} , we can deduce from (3.4) that

$$\begin{aligned} \lambda \int |Du_j(X)|^p dX &\leq \left| \int (\mathfrak{h} - u) d\mu_j - \int \langle \mathcal{A}(X, Du), Dh \rangle dX \right| \\ &\leq \sup_{\partial D} \varphi \cdot \mu_j(\mathbb{R}^n \setminus D) + \Lambda \int |Du_j|^{p-1} |Dh| dX \\ &\leq \sup_{\partial D} \varphi \cdot \mu_j(\mathbb{R}^n \setminus D) + \frac{\lambda}{2} \int |Du_j|^p dX + C_1 \int |Dh|^p dX. \end{aligned} \quad (3.5)$$

In the last step we have used Young's inequality and $C_1 = \epsilon^{-p}/p$ where ϵ satisfies $\epsilon^{p/(p-1)} = p\lambda/2(p-1)$. In view of (3.2) and the estimate in (3.5), we reach the conclusion that there exists a constant C_1 , depending only on \mathcal{A} , D and φ , such that

$$\|\nabla u_j\|_{L^p(D^C)}^p \leq C_1 \left(1 + \frac{1}{2\alpha} \int_{\partial D} \partial_{\mathcal{A}} u_j(X) d\mathcal{H}^{n-1}(X) \right), \quad (3.6)$$

where $\alpha := \mathcal{H}^{n-1}(\partial D)$. From the monotonicity and convexity properties of the non-linearity Γ , we derive, for each $Y \in \partial D$ fixed, that

$$2\Gamma \left(Y, \|\nabla u_j\|_{L^p(D^C)}^p \right) \leq C_2 + \Gamma \left(Y, \frac{1}{\alpha} \int_{\partial D} \partial_{\mathcal{A}} u_j(X) d\mathcal{H}^{n-1}(X) \right),$$

where C_2 is a constant depending only on \mathcal{A} , D , φ and Γ . Once more using the convexity of $\Gamma(Y, \cdot)$, it follows from Jensen's inequality that

$$2\Gamma \left(Y, \|\nabla u_j\|_{L^p(D^C)}^p \right) \leq C_1 + \frac{1}{\alpha} \int_{\partial D} \Gamma(Y, \partial_{\mathcal{A}} u_j(X)) d\mathcal{H}^{n-1}(X). \quad (3.7)$$

Integrate inequality (3.7) with respect to Y over ∂D and taking into account property (iii) of the non-linearity Γ , we derive

$$\int_{\partial D} \Gamma \left(Y, \|\nabla u_j\|_{L^p(D^C)}^p \right) d\mathcal{H}^{n-1}(Y) \leq C_3 \left(1 + \int_{\partial D} \Gamma(X, \partial_{\mathcal{A}} u_j(X)) d\mathcal{H}^{n-1}(X) \right), \quad (3.8)$$

where again C_3 depends only upon \mathcal{A} , D , φ and Γ . Finally, (3.8) and the coercivity of the function

$$t \mapsto \int_{\partial D} \Gamma(X, t) d\mathcal{H}^{n-1}(X),$$

see (2.5), together complete the proof of the Proposition. \square

In view of the energy estimate provided in Proposition 3.3, it becomes natural to investigate the behavior of \mathfrak{J}_λ over weakly convergent sequences in $W^{1,p}$. In this direction we have

Lemma 3.4. *Let $f_j \in H^1(D^C)$ be a sequence of functions satisfying, $\mathbb{L}f_j \geq 0$ in the distributional sense and, for some $\delta > 0$, $\mathbb{L}f_j = 0$ in $D_\delta := \{X \in D^C \mid \text{dist}(X, \partial D) < \delta\}$. Assume f_j converges weakly to f in $W^{1,p}(D^C)$. Then, $\mathbb{L}f \geq 0$ in the distributional sense, $\mathbb{L}f = 0$ in D_δ and furthermore*

$$\mathfrak{J}(f) + \varrho_\lambda(|\{f > 0\}|) \leq \liminf_{j \rightarrow \infty} \left\{ \mathfrak{J}(f_j) + \varrho_\lambda(|\{f_j > 0\}|) \right\}.$$

Proof. The fact that $\mathbb{L}f \geq 0$ in the distributional sense follows easily. In fact, for any nonnegative $\psi \in C_0^1(D^C)$, we have

$$\langle \mathbb{L}f, \psi \rangle := - \int \langle \mathcal{A}(X, Df), D\psi \rangle dX = - \lim_{j \rightarrow \infty} \int \langle \mathcal{A}(X, Df_j), D\psi \rangle dX \geq 0,$$

since, for any j , $\lim_{j \rightarrow \infty} \int \langle \mathcal{A}(X, Df_j), D\psi \rangle dX \leq 0$. A similar computation shows that $\mathbb{L}f = 0$ in D_δ .

Let us turn our attention to the $W^{1,p}$ -weak lower semicontinuity of the functional \mathfrak{J}_λ . Firstly, the volume penalty term of the functional \mathfrak{J}_λ is indeed weak lower semicontinuous, since, up to a subsequence, $f_j(X) \rightarrow f(X)$ for a.e. $X \in D^C$. Thus, by Fatou's Lemma

$$|\{f > 0\}| \leq \liminf_{j \rightarrow \infty} |\{f_j > 0\}|.$$

Since, the penalty factor ϱ_λ is non-decreasing and continuous, there holds

$$\varrho_\lambda(|\{f > 0\}|) \leq \liminf_{j \rightarrow \infty} \varrho_\lambda(|\{f_j > 0\}|),$$

as desired. We now focus our attention on the functional $\mathfrak{J}(v) = \int_{\partial D} \Gamma(X, \partial_A v) d\mathcal{H}^{n-1}(X)$. As in the Calculus of Variations, in order to establish the $W^{1,p}$ -weak lower semicontinuity of \mathfrak{J} , we shall explore the convexity assumption on $\Gamma(X, \cdot)$. Indeed, we start by analyzing functional with piecewise linear potential, i.e., functionals with this particular profile:

$$\mathfrak{F}_m(v) := \int_{\partial D} F_m(X, \partial_A v) d\mathcal{H}^{n-1}(X), \quad (3.9)$$

where F_m is of the form

$$F_m(X, t) = \max_{1 \leq k \leq m} \{B_k(X)t + C_k(X)\}, \quad B_k, C_k \in C(\partial D). \quad (3.10)$$

We then label, for each $k = 1, 2, \dots, m$, the sets

$$\mathcal{D}_k(f) := \{X \in \partial D \mid F_m(X, \partial_A f(X)) = B_k(X)\partial_A f(X) + C_k(X)\}.$$

Thus $\partial D = \bigcup_{k=1}^m \mathcal{D}_k(f)$, and we may assume that $\mathcal{D}_k(f) \cap \mathcal{D}_i(f) = \emptyset$, whenever $k \neq i$. Also, recall that $\mathbb{L}f_j$ and $\mathbb{L}f$ define Radon measures in D^C , and since $f_j \rightharpoonup f$ in $W^{1,p}$, by standard elliptic estimates, we have

$$\mathbb{L}f_j \xrightarrow{*} \mathbb{L}f,$$

in the sense of Radon measures. Therefore, using a representation as in (3.2), we obtain that, for any continuous function $\zeta \in C(\partial D)$,

$$\int_{\partial D} \zeta(X) \partial_A f(X) d\mathcal{H}^{n-1}(X) \leq \liminf_{j \rightarrow \infty} \int_{\partial D} \zeta(X) \partial_A f_j(X) d\mathcal{H}^{n-1}(X).$$

With the above at hands, we estimate

$$\begin{aligned} \mathfrak{F}_m(f) &= \sum_{k=1}^m \int_{\mathcal{D}_k(f)} \{B_k(X)\partial_A f + C_k(X)\} d\mathcal{H}^{n-1}(X) \\ &\leq \liminf_{j \rightarrow \infty} \sum_{k=1}^m \int_{\mathcal{D}_k(f)} \{B_k(X)\partial_A f_j + C_k(X)\} d\mathcal{H}^{n-1}(X) \\ &\leq \liminf_{j \rightarrow \infty} \mathfrak{F}_m(f_j), \end{aligned}$$

In other words, we have proven functionals as in (3.9) are $W^{1,p}$ -weak lower semicontinuous. Finally, under the assumption that $\Gamma(X, \cdot)$ is convex we know that for each $X \in \partial D$ there exists a sequence of functions $F_m(X, t)$ as in (3.10) such that, for any t ,

$$\Gamma(X, t) = \lim_{m \rightarrow \infty} F_m(X, t). \quad (3.11)$$

As a combination of (3.11) and the $W^{1,p}$ -weak lower semicontinuity of each \mathfrak{F}_m , the Lemma follows. \square

The results proven in Proposition 3.3 and in Lemma 3.4 are important piece of information towards establishing the existence of an optimal shape for problem (2.10); however, at this precise stage, those are not enough. We would like to invite the readers to make a small pause in order to appreciate the intrinsic difficulty involved in proving the existence of a minimal configuration to the penalized problem (2.10).

Following the natural scheme, one considers a minimizing sequence, Ω_j , to the functional \mathfrak{J}_λ , i.e.,

$$\mathfrak{J}_\lambda(\Omega_j) \xrightarrow{j \rightarrow \infty} \min_{\Omega \supset D} \mathfrak{J}_\lambda.$$

If u_j denotes the \mathcal{A} -potential associated to the configuration Ω_j , it follows from Proposition 3.3 that, up to a subsequence, u_j converges weakly and almost everywhere to a function $u \in W^{1,p}(D^C)$ which is non-negative. As a consequence of Lemma 3.4, we have that $\mathbb{L}u \geq 0$. Furthermore,

$$\int_{\partial D} \Gamma(X, \partial_{\mathcal{A}} u) d\mathcal{H}^{n-1}(X) + \varrho_\lambda(|\{u > 0\}|) \leq \min_{\Omega \supset D} \mathfrak{J}_\lambda.$$

Therefore, a natural candidate for an optimal shape to problem (2.10) is

$$\Omega := \{X \in \mathbb{R} \setminus D \mid u(X) > 0\}.$$

However, with the information we have so far, it is not possible to guarantee that (Ω, u) is an admissible pair, i.e., that u is the \mathcal{A} -potential associated to Ω , or equivalently that

$$\mathbb{L}u = 0 \text{ in } \Omega.$$

In fact, it is not true, in general, that if an ordinary sequence of functions u_j , satisfying $\Delta u_j = 0$ in $\{u_j > 0\}$, converges weakly in H^1 to u , then $\Delta u = 0$ in $\{u > 0\}$. As a general comment, the above described difficulty is one of the features that makes problems with varying domains (free boundary problems) notably more delicate.

4 Existence of minimizer to problem (\mathfrak{P}_λ)

Well, the scheme presented at the end of the previous section is not pointless: since our sequence is converging to a special configuration, namely a minimizer for the functional \mathfrak{J}_λ , we should keep the hope that this strong additional ingredient will assure that in fact $\mathbb{L}u = 0$ in $\{u > 0\}$. In this section, we will carry this delicate analysis out, which will ultimately allow us to conclude problem (2.13) has always a minimizer. As we will see, even the weak formulation of the penalty version of our primary goal presents rather delicate mathematical issues. This is due in part to the adverse environment generated by the non-linear and degeneracy features of \mathcal{A} , and in part to the non-local structure of the problem. The latter makes local perturbations inefficient, and thus more creativity is needed to furnish appropriate competing configurations.

As for our first result towards the existence of a minimizer for problem (2.13), we will provide an a-priori estimate on the distance from the free boundary to the fixed boundary. This is an important supporting result as it allows to seek for minimizers in a more suitable class of configurations.

However, in order to accomplish such a result, we initially need to study an auxiliary free boundary problem in the spirit of [AC81], which we present now.

Theorem 4.1. *Let \mathcal{O} be a domain in \mathbb{R}^n and $\psi: \mathcal{O} \rightarrow \mathbb{R}$ a nonnegative function. Let \mathcal{A} be a p -degenerate elliptic map and assume $\mathcal{A}(\cdot, \mathcal{O}) \in C^\epsilon$ for all \mathcal{O} . Then, for any constant $\tau > 0$, there exists a minimizer $v = v_\tau$ to the problem*

$$\text{Minimize} \quad \left\{ E_\tau(f) := \int_{\mathcal{O}} \{ \langle \mathcal{A}(X, Df), Df \rangle + \tau \chi_{\{f > 0\}} \} dX \mid f \in W^{1,p}(\mathcal{O}), f|_{\partial \mathcal{O}} = \psi \right\}.$$

Furthermore, v is nonnegative, Lipschitz continuous and nondegenerate away from the free boundary $\partial\{v_\tau > 0\}$.

With the free boundary technology available nowadays, it is not hard to establish the existence as well as optimal regularity and nondegeneracy of a minimizer to the above problem. Basically there are two procedures that lead to these results: one can directly approach the minimization problem, by mixing the strategy as in [AC81] and [DP05]. Another charming and fruitful strategy

is to employ a regularizing technique method, basically by mixing the estimates in [MT07] or [Teix1] and [DPS03], see also [K06]. Mathematically the latter is described as follows: choose your favorite nonnegative bounded real function β , such that $\text{supp}\beta = [0, 1]$ and, say, $\int_0^1 \beta(\zeta) d\zeta = 1$. For each $\varepsilon > 0$ define

$$\beta_\varepsilon(t) := \frac{1}{\varepsilon} \beta\left(\frac{t}{\varepsilon}\right),$$

and finally put $B_\varepsilon(s) := \int_0^s \beta_\varepsilon(\zeta) d\zeta$. The ε regularizing problem then becomes

$$\text{Minimize } \left\{ E_\tau^\varepsilon(f) := \int_{\mathcal{O}} \{ \langle \mathcal{A}(X, Df), Df \rangle + \tau B_\varepsilon(f) \} dX \mid f \in W^{1,p}(\mathcal{O}), f|_{\partial\mathcal{O}} = \psi \right\}, \quad (4.1)$$

The existence of minimizers v^ε of (4.1) is standard. One then proves Lipschitz regularity and nondegeneracy for v^ε , uniform in ε . By letting $\varepsilon \searrow 0$, up to a subsequence, v^ε will converge to a locally Lipschitz function v that is a minimizer of E_τ . We omit the details of the proof of Theorem 4.2.

Proposition 4.2. *There exists a positive constant $\gamma > 0$, depending only on dimension, λ , ∂D , Γ and φ such that any (possible) minimizer u^* of problem (2.13) satisfies*

$$D_\gamma := \{X \in D^C \mid \text{dist}(X, \partial D) \leq \gamma\} \subset \{u^* > 0\}.$$

Proof. Let $P \in \partial D$ be fixed and $B = B_r(Y) \subset D$ satisfy

$$\overline{B} \cap \partial D = \{P\}.$$

By a compactness argument on ∂D , we can select an $r < 5\delta_0$, where δ_0 is the universal number from (2.6), such that the above holds for a.e. $P \in \partial D$. In view of Theorem 4.2, there exists a minimization, $v = v(\tau)$ to

$$\text{Min } \left\{ \begin{array}{l} E_\tau(f) := \int_{5B \setminus B} \{ \langle \mathcal{A}(X, Df), Df \rangle + \tau \chi_{\{f>0\}} \} dX \mid f \in W^{1,p}(5B \setminus B), \\ f|_{\partial 5B} = 0, \quad \text{and} \quad f|_{\partial B} = \inf_{\partial D} \varphi \end{array} \right\}. \quad (4.2)$$

Here $\tau > 0$ is a constant to be chosen later. For future reference, let us label the following the sets

$$\Theta := \{X \in D^C \cap 5B \mid v(X) > u(X)\} \quad \text{and} \quad \mathcal{O} := \{X \in D^C \cap 5B \mid v(X) > 0\}.$$

It is important to keep in mind that, from the properties of v , we can ensure that there exist constants $\theta, \hat{\delta} > 0$, depending only on \mathcal{A} , τ , ∂D and $\inf \varphi$ such that

$$|\mathcal{O} \cap D^C| > \theta(\tau), \quad \text{and} \quad \text{dist}(P, (\partial\mathcal{O} \cap D^C)) > \hat{\delta}(\tau). \quad (4.3)$$

We now define the function $m: 5B \setminus B \rightarrow \mathbb{R}_+$ as

$$m(X) := \begin{cases} v(X) & \text{in } (D \setminus B) \cap 5B \\ \min\{u(X), v(X)\} & \text{in } D^C \cap 5B, \end{cases}$$

Since m competes with v in the minimization problem (4.2), we have $E_\tau(v) \leq E_\tau(m)$. Hence, the following inequality holds

$$\int_{\Theta} \langle \mathcal{A}(X, Du), Du \rangle dX - \int_{\Theta} \langle \mathcal{A}(X, Dv), Dv \rangle dX \geq \tau \{ \mathcal{L}^n(\mathcal{O}) - \mathcal{L}^n(\{u > 0\} \cap \mathcal{O}) \} \quad (4.4)$$

Our strategy now is to obtain a competing inequality to (4.4). To this end, let us consider the function $\mathfrak{M}: D^C \rightarrow \mathbb{R}_+$ defined as

$$\mathfrak{M}(X) := \max\{v(X), u(X)\},$$

and compare it with u in terms of the minimization problem (2.13). Using the minimality of u , we obtain

$$\begin{aligned} \varrho_\lambda \left(\mathcal{L}^n(\mathcal{O}) + \mathcal{L}^n(\{u > 0\}) - \mathcal{L}^n(\{u > 0\} \cap \mathcal{O}) \right) &= \varrho_\lambda \left(\mathcal{L}^n(\{u > 0\}) \right) \\ &\geq \int_{\partial D} \Gamma(X, \partial_{\mathcal{A}} u) - \Gamma(X, \partial_{\mathcal{A}} \mathfrak{M}) d\mathcal{H}^{n-1}(X). \end{aligned} \quad (4.5)$$

From properties 1 and 2 of Γ , and the Lipschitz continuity of the penalty term ϱ_λ , we conclude from (4.5) that there exists a small constant $\alpha_0 = \alpha_0(\partial D, \Gamma)$ such that

$$\frac{\lambda}{\alpha_0} (\mathcal{L}^n(\mathcal{O}) - \mathcal{L}^n(\{u > 0\} \cap \mathcal{O})) \geq \int_{\partial D} \{\partial_{\mathcal{A}} u - \partial_{\mathcal{A}} \mathfrak{M}\} d\mathcal{H}^{n-1}(X). \quad (4.6)$$

Applying the Divergence Theorem (see the representation in (3.1)) and taking into account that $v(X)\mathbb{L}v(X) = 0$ a.e., we obtain

$$\int_{\partial D} \{\partial_{\mathcal{A}} u - \partial_{\mathcal{A}} \mathfrak{M}\} d\mathcal{H}^{n-1}(X) \geq \frac{1}{\sup_{\partial D} \varphi} \int_{\Theta} \langle ADu, Du \rangle - \langle ADv, Dv \rangle dX. \quad (4.7)$$

As a combination of (4.4), (4.6) and (4.7) we deduce that

$$\sup_{\partial D} \varphi \cdot \frac{\lambda}{\alpha_0} \left[\mathcal{L}^n(\mathcal{O}) - \mathcal{L}^n(\{u > 0\} \cap \mathcal{O}) \right] \geq \tau \left[\mathcal{L}^n(\mathcal{O}) - \mathcal{L}^n(\{u > 0\} \cap \mathcal{O}) \right].$$

Thus, if τ is chosen big enough, depending only upon dimension, \mathcal{A} , ∂D and φ , there must be the case that

$$\mathcal{O} \subset \{u^* > 0\}.$$

This together with (4.3) ultimately finishes the proof of the Proposition. \square

In order to advance in our analysis, we need another related free boundary problem: an \mathcal{A} -obstacle type problem, which again, with the free boundary technology available, is easy accomplished and therefore we omit the details.

Theorem 4.3. *Let \mathcal{M} be a measurable set in D^C . There exists a unique function \mathfrak{b} , solution to the following obstacle-type problem:*

$$\text{Min} \left\{ \int_{D^C} \langle \mathcal{A}(X, Df), Df \rangle dX \mid f \in W^{1,p}(D^C) \text{ } f = \varphi \text{ on } \partial D \text{ and } f \leq 0 \text{ in } \mathcal{M} \right\}.$$

Furthermore, $\sup \varphi \geq \mathfrak{b} \geq 0$, $\mathbb{L}\mathfrak{b} = 0$ in $\{\mathfrak{b} > 0\}$ and $\int \mathfrak{b} \mathbb{L}\mathfrak{b} dX = 0$.

We now can state and proof our main theorem concerning the existence of an optimal configuration to weak formulations of problem (2.4), namely problems $(\mathfrak{P}_\lambda^{\text{weak}})$ and (\mathfrak{P}_λ) .

Theorem 4.4. *There exists an optimal configuration Ω_λ^* to problem (2.10) (the penalized problem (\mathfrak{P}_λ)). Furthermore, for a universal modulus of continuity σ , the \mathcal{A} -potential associated to Ω_λ^* , u_λ^* , is σ -continuous in D^C and $\|u_\lambda^*\|_{C^\sigma} \lesssim K(\lambda, D, \varphi, \Gamma, \mathcal{A})$.*

Proof. Before starting the proof, let us explain its strategy. We will initially establish the existence of a minimizer to a very weak formulation to problem (2.10). Afterwards we “regularize” the minimizer we have found via a stabilization phenomenon. Here are the details: Lemma 3.4 assures, for each $0 < \delta \ll 1$, the existence of a function $u_\lambda^\delta \in \mathcal{V}(\delta)$, satisfying

$$\mathfrak{J}_\lambda(u_\lambda^\delta) = \min_{V(\delta)} \mathfrak{J}_\lambda.$$

Furthermore, by noticing that the same computation employed in the proof of Theorem 4.2 applies if we restrict ourselves to configurations in $\mathcal{V}(\delta)$, we know that

$$D_\gamma \subset \{u_\lambda^\delta > 0\}, \quad \forall \delta > 0.$$

Let $B = B_r(X_0)$ be a fixed ball in D^C and \mathfrak{b} be the solution provided by Theorem 4.3 to

$$\text{Min} \left\{ \int_{D^C} \langle \mathcal{A}(X, Df), Df \rangle dX \mid f \in W^{1,p}(D^C) \text{ } f = \varphi \text{ on } \partial D \text{ and } f \leq 0 \text{ in } \{u_\lambda^\delta = 0\} \setminus B \right\}. \quad (4.8)$$

We also consider \mathfrak{h} to be the \mathcal{A} -harmonic function in B that agrees with u_λ^δ on B^C . It is standard to verify that

$$0 \leq u \leq \mathfrak{b} \leq \mathfrak{h} \leq \sup \varphi. \quad (4.9)$$

As before, (more precisely, as in the proof of Proposition 4.2) taking into account that $\int \mathfrak{b} \mathfrak{L} \mathfrak{b} dX = 0$, we find

$$\int_{\partial D} \Gamma(X, \partial_{\mathcal{A}} u) - \Gamma(X, \partial_{\mathcal{A}} \mathfrak{b}) \geq c_1 \left(\int_{D^C} \langle \mathcal{A}(X, Du_\lambda^\delta), Du_\lambda^\delta \rangle dX - \int_{D^C} \langle \mathcal{A}(X, D\mathfrak{b}), D\mathfrak{b} \rangle dX \right), \quad (4.10)$$

for a universal positive constant $c_1 > 0$. However, \mathfrak{h} competes with \mathfrak{b} in the obstacle problem (4.8), thus, (4.10) becomes

$$\int_{\partial D} \Gamma(X, \partial_{\mathcal{A}} u) - \Gamma(X, \partial_{\mathcal{A}} \mathfrak{b}) \geq c_1 \left(\int_{D^C} \langle \mathcal{A}(X, Du_\lambda^\delta), Du_\lambda^\delta \rangle dX - \int_{D^C} \langle \mathcal{A}(X, D\mathfrak{h}), D\mathfrak{h} \rangle dX \right). \quad (4.11)$$

For the moment, let us assume $p \geq 2$. If we take into account Lemma 3.2, we can enhance the estimate by below in (4.11) as

$$\int_{\partial D} \Gamma(X, \partial_{\mathcal{A}} u) - \Gamma(X, \partial_{\mathcal{A}} \mathfrak{b}) \geq c_2 \left(\int_{D^C} |\nabla (u_\lambda^\delta - \mathfrak{h})(X)|^p dX \right), \quad (4.12)$$

for an appropriate positive but small constant c_2 . Our next step is to compare u_λ^δ and \mathfrak{b} in terms of the functional \mathfrak{J}_λ . By doing so, in view of (4.12), we obtain

$$\lambda \mathcal{L}^n(\{X \in B_r(X_0) \mid u_\lambda^\delta(X) = 0\}) \geq c_3 \left(\int_{D^C} |\nabla (u_\lambda^\delta - \mathfrak{h})(X)|^p dX \right), \quad (4.13)$$

for another constant $c_3 > 0$, depending on dimension, \mathcal{A} , $\sup \varphi$, and Γ . If $1 < p \leq 2$, we obtain

$$[\lambda \mathcal{L}^n(\{X \in B_r(X_0) \mid u_\lambda^\delta(X) = 0\})]^{p/2} \times \left[\int_{D^C} |\nabla u_\lambda^\delta|^p dX \right]^{1-\frac{p}{2}} \geq c_3 \left(\int_{D^C} |\nabla (u_\lambda^\delta - \mathfrak{h})(X)|^p dX \right). \quad (4.14)$$

In any case, our conclusion is that if $B_r(X_0) \subset D_\gamma$, then $|\{X \in B_r(X_0) \mid u_\lambda^\delta(X) = 0\}| = 0$ and consequently, from either (4.13) or (4.14), u_λ^δ is \mathcal{A} -harmonic there. Of course $\mathfrak{J}(u_\lambda^{\delta_1}) \leq \mathfrak{J}(u_\lambda^{\delta_2})$, provided $\delta_1 \leq \delta_2$. However, from the fact that $\mathfrak{L}u_\lambda^\delta = 0$ in D_γ we have a much stronger conclusion:

$$\mathfrak{J}(u_\lambda^{\delta_1}) = \mathfrak{J}(u_\lambda^{\delta_2}),$$

whenever $\delta_1, \delta_2 \leq \gamma$. We have proven the existence of a minimizer u_λ^* to $(\mathfrak{P}_\lambda^{\text{weak}})$, that is, problem (2.13).

Our next step is now to prove that $\Omega^* := \{u_\lambda^* > 0\}$ is a minimizer to problem (2.10). For that, we have to show

$$\mathcal{L}u_\lambda^* = 0 \text{ in } \Omega^*.$$

Well, but again it is a standard argument to show from either (4.13) or (4.14) that u_λ^* belongs to an appropriate De Giorgi's class (recall \mathfrak{h} is Hölder continuous by elliptic estimates). Therefore, there indeed exists a modulus of continuity $\sigma(\sigma(t) = |t|^\alpha, \text{ for some } \alpha > 0)$, such that

$$|u(X) - u(Y)| \leq C\lambda\sigma(|X - Y|).$$

In order to prove that $\mathcal{L}u = 0$ in $\{u > 0\}$, we argue as follows: let $X_0 \in \{u > 0\}$ be a generic point. By the continuity of u , there exists an $r_0 > 0$ such that $B_{r_0}(X_0) \subset \{u > 0\}$. Therefore, in view of (4.13) or (4.14), we conclude, as before that

$$u = \mathfrak{h} \text{ in } B_{r_0}(X_0),$$

and the Theorem is finally proven. \square

5 Existence of an optimal shape to problem (2.4) in low dimensions

In this section, upon a technical restriction on the dimension, we will show that the original volume constrained problem (2.4) admits an optimal configuration. The theory that addresses the existence of an optimal design for problem (2.4) in all dimensions will be developed in section 7.

Our strategy is based on a limiting analysis on the penalized problem (2.10). For that, we initially need a simple lemma.

Lemma 5.1. *There exists a constant $C > 0$, depending on \mathcal{A}, Γ, D and φ , but independent of λ , such that if u_λ^* is the \mathcal{A} -potential associated to an optimal shape Ω_λ^* for problem (2.10), then*

$$\int_{D^c} |\nabla u_\lambda^*(X)|^p dX < C.$$

Proof. Let \mathcal{O} be your favorite smooth configuration surrounding D that satisfies

$$\mathcal{L}^n(\mathcal{O} \setminus D) = \iota,$$

and let ω be its \mathcal{A} -potential, i.e., the \mathcal{A} -harmonic function in $\mathcal{O} \setminus D$ taking φ and 0 as boundary data on ∂D and $\partial \mathcal{O}$ respectively. By the minimality property of Ω_λ^* , we know

$$\begin{aligned} \int_{\partial D} \Gamma(X, \partial_{\mathcal{A}} u_\lambda^*) d\mathcal{H}^{n-1}(X) &\leq \mathfrak{J}_\lambda(\Omega_\lambda^*) \\ &\leq \mathfrak{J}_\lambda(\mathcal{O}) \\ &= \int_{\partial D} \Gamma(X, \partial_{\mathcal{A}} \omega) d\mathcal{H}^{n-1}(X) \\ &= \overline{C}_0, \end{aligned} \tag{5.1}$$

where \overline{C}_0 is universal, as it depends only on your choice for \mathcal{O} . On the other hand, using the results and notations of Lemma 3.1, we have

$$\begin{aligned} \int_{\mathbb{R}^n \setminus D} \langle \mathcal{A}(X, Du_\lambda^*), Du_\lambda^* \rangle dX &= \int_{\mathbb{R}^n \setminus D} u_\lambda^*(X) d\mu_{u_\lambda^*}(X) \\ &\leq \sup_{\partial D} \varphi \cdot \mu_{u_\lambda^*}(\mathbb{R}^n \setminus D) \\ &= \sup_{\partial D} \varphi \cdot \int_{\partial D} \partial_{\mathcal{A}} u_\lambda^*(S) d\mathcal{H}^{n-1}(S). \end{aligned} \tag{5.2}$$

From ellipticity and (5.2), we conclude

$$\underline{c}_1 \int_{D^C} |\nabla u_\lambda^*(X)|^p dX \leq \frac{1}{\mathcal{H}^{n-1}(\partial D)} \int_{\partial D} \partial_{\mathcal{A}} u_\lambda^*(S) d\mathcal{H}^{n-1}(S), \quad (5.3)$$

where \underline{c}_1 is a positive number that depends on \mathcal{A} , φ and D . Now, for each $Y \in \partial D$ fixed, we obtain from (5.3)

$$\begin{aligned} \Gamma\left(Y, \underline{c}_1 \int_{D^C} |\nabla u_\lambda^*(X)|^p dX\right) &\leq \Gamma\left(Y, \frac{1}{\mathcal{H}^{n-1}(\partial D)} \int_{\partial D} \partial_{\mathcal{A}} u_\lambda^*(S) d\mathcal{H}^{n-1}(S)\right) \\ &\leq \frac{1}{\mathcal{H}^{n-1}(\partial D)} \int_{\partial D} \Gamma(Y, \partial_{\mathcal{A}} u_\lambda^*(S)) d\mathcal{H}^{n-1}(S). \end{aligned} \quad (5.4)$$

In the last inequality we have used Jensen's Theorem. If we integrate (5.4) with respect to Y over ∂D , we reach the following conclusion

$$\int_{\partial D} \Gamma\left(Y, \underline{c}_1 \int_{D^C} |\nabla u_\lambda^*(X)|^p dX\right) d\mathcal{H}^{n-1}(Y) \leq \overline{C}_2 \int_{\partial D} \Gamma(X, \partial_{\mathcal{A}} u_\lambda^*) d\mathcal{H}^{n-1}(X), \quad (5.5)$$

where \overline{C}_2 depends only on ∂D and the non-linearity Γ . Finally, if we combine (5.1), (5.5) and (2.5), we deduce that there must exist a constant $C > 0$ depending only on \mathcal{A} , Γ , D and φ , such that

$$\int_{D^C} |\nabla u_\lambda^*(X)|^p dX \leq C, \quad (5.6)$$

which is precisely the thesis of the Lemma. \square

Theorem 5.2. *Assume the dimension n is less than p . Then there exists an optimal configuration Ω^* to problem (2.4).*

Proof. Because of Lemma (5.1), up to a subsequence, we can assume u_λ converges, as $\lambda \rightarrow \infty$, weakly in $W^{1,p}(D^C)$ to a function u^* . Furthermore, since we have assumed $n < p$, it follows by the classical Sobolev Imbedding (see, for instance, [Adams75]), that passing to another subsequence if necessary, we can further assume that u_λ converges locally uniformly to u^* in $\mathbb{R}^n \setminus D$ and thus, u^* is continuous in D^C . We claim that

$$\mathbb{L}u^* = 0 \text{ in } \Omega^* := \{X \in D^C \mid u^*(X) > 0\}.$$

Indeed, let $X_0 \in \Omega^*$ be an arbitrary point in the set of positivity of u^* , say $u^*(X_0) = \delta_0 > 0$. By continuity, there exists an $r_0 > 0$ such that

$$u^*(X) > \frac{\delta_0}{3} \text{ in } B_{r_0}(X_0).$$

Since u_λ^* converges uniformly to u^* in $B_{r_0}(X_0)$, there exists a λ_0 large enough, such that

$$u_\lambda^*(X) > \frac{\delta_0}{7} \text{ in } B_{r_0}(X_0), \quad \forall \lambda > \lambda_0.$$

However, we have proven that $\mathbb{L}u_\lambda^* = 0$ in $\{u_\lambda^* > 0\}$. Therefore, for λ large enough, each u_λ^* is \mathcal{A} -harmonic in $B_{r_0}(X_0)$. Thus, as argued in the proof of Lemma 3.4, we in fact conclude u^* is \mathcal{A} -harmonic in its set of positivity and the first claim is proven.

Notice furthermore that, in view of Proposition 4.2,

$$\text{dist}(\partial D, \partial \Omega^*) > \gamma,$$

for some $\gamma > 0$. From inequality (5.1), we have, in particular, that

$$\lambda (\mathcal{L}^n(\Omega_\lambda^* \setminus D) - \iota)^+ \leq \overline{C}_0,$$

for a universal constant \overline{C}_0 . Thus, using Fatou's Lemma we see that

$$\begin{aligned} (\mathcal{L}^n(\Omega^* \setminus D) - \iota)^+ &\leq \liminf_{\lambda \rightarrow \infty} (\mathcal{L}^n(\Omega_\lambda^* \setminus D) - \iota)^+ \\ &= 0. \end{aligned}$$

That is, our candidate to an optimal design for problem (2.4), Ω^* , does satisfy

$$\mathcal{L}^n(\Omega^* \setminus D) \leq \iota,$$

so it competes in problem (2.4). Our final step is to show that in fact Ω^* is an optimal configuration for problem (2.4). For that, let \mathfrak{C} be any competing configuration for problem (2.4), i.e., $\mathcal{L}^n(\mathfrak{C} \setminus D) \leq \iota$, and v its \mathcal{A} -potential, that is, v satisfies

$$\mathbb{L}v = 0 \text{ in } \mathfrak{C} \setminus D, \quad v = \varphi \text{ on } \partial D, \quad v = 0 \text{ on } \partial \mathfrak{C}.$$

In particular \mathfrak{C} competes with u_λ^* in (\mathfrak{P}_λ) , problem (2.10); therefore,

$$\begin{aligned} \mathfrak{J}(\mathfrak{C}) &:= \int_{\partial D} \Gamma(X, \partial_{\mathcal{A}} v(X)) d\mathcal{H}^{n-1}(X) \\ &= \mathfrak{J}_\lambda(\mathfrak{C}) \\ &\geq \mathfrak{J}_\lambda(\Omega_\lambda^*) \\ &\geq \int_{\partial D} \Gamma(X, \partial_{\mathcal{A}} u_\lambda^*(X)) d\mathcal{H}^{n-1}(X) \\ &\geq \int_{\partial D} \Gamma(X, \partial_{\mathcal{A}} u^*(X)) d\mathcal{H}^{n-1}(X) + O(1), \end{aligned}$$

because of the weak lower semicontinuity feature of \mathfrak{J} proven in Lemma 3.4. Finally if we let $\lambda \rightarrow \infty$ in the above chain of inequalities, the Theorem is proven. \square

It is worth to point out that Theorem 5.2 gives the existence of an optimal configuration to problem (2.4) with no regularity whatsoever on the medium. That is, up to this point of the project, the operator \mathcal{A} has been a general bounded measurable degenerated elliptic map. However, it turns out that in order to advance on the study of existence of optimal shapes for problem (2.4), with no restriction on the dimension, some extra information is needed to perform appropriate perturbations on the optimal designs Ω_λ^* . This will be the contents of the next two sections.

6 Continuous medium and fine weak geometric properties of the free boundary

In this section we will prove that the free boundary, $\partial\Omega_\lambda^*$ enjoys the appropriate weak geometry. This feature will allow us to produce geometric-measures perturbations that will ultimately lead us to conclude that, if the penalty term λ is too large, but still finite, then Ω_λ^* , in fact, obey $\mathcal{L}^n(\Omega_\lambda^* \setminus D) \leq \iota$. The latter will be carried out in section 7.

As highlighted in the last paragraph of the previous section, in order to accomplish a deeper understanding on the free boundary $\partial\Omega_\lambda^*$, we will need to enforce a mild continuity assumption on the medium. Thus, hereafter, unless otherwise stated, we shall assume that for some $\epsilon > 0$, the map

$$X \mapsto \mathcal{A}(X, \xi) \in C^\epsilon(\mathbb{R}^n \setminus D), \quad \forall \xi \in \mathbb{R}^n. \quad (6.1)$$

Mathematically, condition (6.1) enables $C^{1,\alpha}$ elliptic estimates for solutions to

$$\mathbb{L}\psi = 0$$

and, at least equally important, it unlocks the Hopf's maximum principle for \mathcal{A} -harmonic functions.

As for our first theorem in this section, we will obtain optimal regularity for \mathcal{A} -potentials u_λ^* associated to optimal configurations Ω_λ^* of Problem (\mathfrak{P}_λ) , that is, Problem (2.10). Notice that inside Ω_λ^* , the function u_λ^* satisfies $\mathcal{L}u_\lambda^*$; therefore, it is locally $C^{1,\alpha}$ smooth. However, from the Hopf's maximum principle, u_λ^* reaches the free boundary with a positive slope, thus ∇u_λ^* jumps from a positive value to zero through the free boundary, $\partial\Omega_\lambda^*$. The conclusion is that the optimal regularity we can hope for u_λ^* is Lipschitz continuity. This is the contents of the next Theorem.

Theorem 6.1. *Let Ω_λ^* be an optimal configuration to Problem (2.10) and u_λ^* its \mathcal{A} potential. Then,*

$$\|\nabla u_\lambda^*\|_{L^\infty(\mathbb{R}^n \setminus D)} \leq C\lambda^{1/p},$$

for a constant C that depends only on \mathcal{A} , Γ , φ and D .

Proof. We will provide two proofs of this important theorem. The first one follows the glamorous approach suggested in [AC81]. Unfortunately, for non-local problems like ours, the efficiency of that method is restricted to the case $p \geq 2$ and a new and more modern argument is required to establish Lipschitz continuity for \mathcal{A} -potential associated to an optimal design with when $1 < p < 2$. The second proof we will present works for all $p > 1$.

1st Proof. The case $p \geq 2$. We shall initially obtain an competing estimate for Inequality (4.13), with u_λ^δ replaced by u_λ^* . Enhancing the notation in the proof of Theorem 4.4, $B = B_d(X_0)$ will be a ball centered at a point in Ω_λ^* , $\text{dist}(X_0, \partial D) \gg d \geq \text{dist}(X_0, \partial\Omega_\lambda^*)$ and \mathfrak{h} the \mathcal{A} -harmonic function in B that agrees with u_λ^* on ∂B . For any direction $\nu \in \mathbb{S}^{n-1}$, we define

$$r_\nu := \min \left\{ r \mid \frac{1}{4} \leq r \leq 1 \text{ and } u_\lambda^*(X_0 + dr\nu) = 0 \right\}$$

if such a set is nonempty; otherwise, we put $r_\nu = 1$. For almost every direction ν the map $r \mapsto u_\lambda^*(X_0 + dr\nu)$ is in $W^{1,p}[\frac{1}{4}, 1]$. Thus, taking into account that $u_\lambda^*(X_0 + dr\nu) = 0$ whenever $r_\nu < 1$, we can compute,

$$\begin{aligned} \mathfrak{h}(X_0 + dr_\nu\nu) &= \int_{r_\nu}^1 \frac{d}{dr} (u_\lambda^* - \mathfrak{h})(X_0 + dr\nu) dr \\ &\leq d \cdot (1 - r_\nu)^{1/p'} \times \left[\int_{r_\nu}^1 |\nabla(\mathfrak{h} - u_\lambda^*)(X_0 + r\nu)|^p dr \right]^{1/p}, \end{aligned} \quad (6.2)$$

where, as usual, p' denotes the conjugate of p , i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. Now, by the Harnack Inequality, we know

$$\inf_{B_{\frac{7}{8}}} \mathfrak{h} \geq c_1 \mathfrak{h}(X_0), \quad (6.3)$$

for a constant $c_1 > 0$ that depends only on dimension and \mathcal{A} . Here $B_{\frac{7}{8}}$ stands for $B_{\frac{7}{8}d}(X_0)$. Let us consider the universal barrier, \mathfrak{B} , given by

$$\begin{cases} \text{div}(\mathcal{A}(X_0 + dX, D\mathfrak{B}(X))) &= 0 \text{ in } B_1(0) \setminus B_{\frac{7}{8}}(0) \\ \mathfrak{B} &= 0 \text{ on } \partial B_1(0) \\ \mathfrak{B} &= c_1 \text{ in } \overline{B_{\frac{7}{8}}(0)}, \end{cases} \quad (6.4)$$

where c_1 is the universal constant in (6.3). By the Hopf's maximum principle, there exists a universal constant $c_2 > 0$, such that

$$\mathfrak{B}(X) \geq c_2 (1 - |X|). \quad (6.5)$$

By the maximum principle and (6.5) we can write

$$\mathfrak{h}(X_0 + dX) \geq \mathfrak{h}(X_0) \cdot \mathfrak{B}(X) \geq c_2 \mathfrak{h}(X_0) \cdot (1 - |X|). \quad (6.6)$$

Combining (6.2) and (6.6) we end up with

$$d^p \cdot \left[\int_{r_\nu}^1 |\nabla(\mathfrak{h} - u_\lambda^*)(X_0 + r\nu)|^p dr \right] \geq c_3 \mathfrak{h}^p(X_0) \cdot (1 - r_\nu). \quad (6.7)$$

Integrating (6.7) with respect to ν over \mathbb{S}^{n-1} , taking into account the definition of r_ν , we find

$$\left(\frac{\mathfrak{h}(X_0)}{d} \right)^p \cdot \int_{B_d(X_0) \setminus B_{d/4}(X_0)} \chi_{\{u_\lambda^*=0\}} dX \leq C_4 \int_{B_d(X_0)} |\nabla(\mathfrak{h} - u_\lambda^*)(X)|^p dX. \quad (6.8)$$

If we replace, in all of our arguments so far, $B_{d/4}(X_0)$ by $B_{d/4}(\overline{X})$, for any $\overline{X} \in \partial B_{d/2}(X_0)$, we obtain

$$\left(\frac{\mathfrak{h}(X_0)}{d} \right)^p \cdot \int_{B_d(X_0) \setminus B_{d/4}(\overline{X})} \chi_{\{u_\lambda^*=0\}} dX \leq \tilde{C}_4 \int_{B_d(X_0)} |\nabla(\mathfrak{h} - u_\lambda^*)(X)|^p dX, \quad \forall \overline{X} \in \partial B_{d/2}(X_0). \quad (6.9)$$

Integrating (6.9) with respect to \overline{X} , we prove the following important estimate:

$$\left(\frac{\mathfrak{h}(X_0)}{d} \right)^p \cdot |\{X \in B_d(X_0) \mid u_\lambda^*(X) = 0\}| \leq C_5 \int_{B_d(X_0)} |\nabla(\mathfrak{h} - u_\lambda^*)(X)|^p dX. \quad (6.10)$$

Now we argue as follows: let $\rho := \text{dist}(X_0, \partial\Omega)$ and for each $0 < \delta \ll 1$, denote \mathfrak{h}_δ the \mathcal{A} -harmonic function in $B_{\rho+\delta}(X_0)$ that agrees with u_λ^* on $\partial B_{\rho+\delta}(X_0)$. Combining (4.13) and (6.10) together with standard elliptic estimate, we deduce

$$\begin{aligned} u_\lambda^*(X_0) &= \mathfrak{h}_\delta(X_0) + O(1) \\ &\leq C\lambda^{1/p}(\rho + \delta) + O(1). \end{aligned} \quad (6.11)$$

Letting $\delta \searrow 0$ in (6.11) we finally conclude

$$u_\lambda^*(X_0) \leq C \text{dist}(X_0, \partial\Omega_\lambda^*),$$

which clearly implies that u_λ^* is Lipschitz continuous up to the free boundary $\partial\Omega_\lambda^*$ and $\|\nabla u_\lambda^*\|_\infty \lesssim \lambda^{1/p}$.

2nd Proof. The general case. Let us assume, for purpose of contradiction, that there exists a sequence of points $X_k \in \Omega_\lambda^*$, with

$$X_k \rightarrow \partial\Omega_\lambda^*, \quad \text{and} \quad \frac{u_\lambda^*(X_k)}{\text{dist}(X_k, \Omega_\lambda^*)} \nearrow +\infty.$$

For convenience, we will call $N_k := u(X_k)$ and $d_k := \text{dist}(X_k, \Omega_\lambda^*)$, thus our assumption is that

$$\frac{d_k}{N_k} = O(1). \quad (6.12)$$

For each k , let Y_k be a point on $\partial\Omega_\lambda^*$ that satisfies

$$|Y_k - X_k| = d_k.$$

By replacing X_k by another point \tilde{X}_k , if necessary, because of the weak maximum principle we can assume that

$$N_k = \sup_{B_{d_k}(Y_k)} u. \quad (6.13)$$

On the other hand, by the Harnack inequality, there exists a universal constant $\kappa > 0$, for which, $\inf_{B_{\frac{2}{3}d_k}} u \geq \kappa N_k$. Thus

$$\sup_{B_{\frac{1}{3}d_k}(Y_k)} u \geq \kappa N_k. \quad (6.14)$$

Now for each $\frac{1}{3} \leq \gamma < 1$, let \mathfrak{h}_γ be the \mathbb{L} -harmonic function in $B_{\gamma d_k}(Y_k)$ taking boundary data equals u_λ^* . By comparing, in terms of the optimal design problem (2.10), u_λ^* and the solution to the Obstacle problem in Theorem 4.3 with $\mathcal{M} = \{u_\lambda^* = 0\} \setminus B_{\gamma d_k}(Y_k)$, we deduce, as in the proof of Theorem 4.4, that

$$\left(\int_{B_{\gamma d_k}(Y_k)} \langle \mathcal{A}(X, Du_\lambda^*), Du_\lambda^* \rangle - \langle \mathcal{A}(X, D\mathfrak{h}_\gamma), D\mathfrak{h}_\gamma \rangle dX \right) \leq \lambda (\gamma d_k)^n. \quad (6.15)$$

For each $k \geq 1$, we consider the functions $\mathcal{U}_k^\gamma, \mathcal{H}_k^\gamma : B_1 \rightarrow (0, 1)$ given by

$$\mathcal{U}_k^\gamma(Z) := \frac{1}{N_k} u_\lambda^*(Y_k + \gamma d_k Z) \quad \text{and} \quad \mathcal{H}_k^\gamma(Z) := \frac{1}{N_k} \mathfrak{h}(Y_k + \gamma d_k Z). \quad (6.16)$$

From (6.14), we know that

$$\sup_{B_{\frac{1}{3}\gamma}} \mathcal{U}_k^\gamma \geq \kappa. \quad (6.17)$$

We also know that \mathcal{H}_k^γ is the unique minimizer of

$$\mathcal{D}(v) := \int_{B_1} \langle \mathcal{A}(Y_k + \gamma d_k X, Dv), Dv \rangle dX, \quad (6.18)$$

among functions $v \in W_0^{1,p}(B_1) + \mathcal{H}_k^\gamma$ and it satisfies

$$\begin{cases} \operatorname{div}(\mathcal{A}(Y_k + \gamma d_k, D\mathcal{H}_k^\gamma)) &= 0 & \text{in } B_1 \\ \mathcal{H}_k^\gamma &= \mathcal{U}_k^\gamma & \text{on } \partial B_1. \end{cases} \quad (6.19)$$

A direct computation reveals that

$$\nabla \mathcal{U}_k^\gamma(Z) = \frac{\gamma d_k}{N_k} \nabla u_\lambda^*(Y_k + \gamma d_k Z) \quad \text{and similarly} \quad \nabla \mathcal{H}_k^\gamma(Z) = \frac{\gamma d_k}{N_k} \nabla \mathfrak{h}(Y_k + \gamma d_k Z). \quad (6.20)$$

Combining (6.20) and the Change of Variables Theorem we obtain that

$$\int_{B_{\gamma d_k}(Y_k)} \langle \mathcal{A}(X, Du_\lambda^*), Du_\lambda^* \rangle dX = \left\{ \frac{\gamma d_k}{N_k} \right\}^{-p} \cdot (\gamma d_k)^n \int_{B_1} \langle \mathcal{A}(Y_k + \gamma d_k, D\mathcal{U}_k^\gamma), D\mathcal{U}_k^\gamma \rangle dX, \quad (6.21)$$

and the same holds when we replace u_λ^* by \mathfrak{h}_γ and \mathcal{U}_k^γ by \mathcal{H}_k^γ . In particular, taking into account (6.12), (6.15), we find that

$$\left(\int_{B_1} \langle \mathcal{A}(Y_k + \gamma d_k, D\mathcal{U}_k^\gamma), D\mathcal{U}_k^\gamma \rangle - \langle \mathcal{A}(Y_k + \gamma d_k, D\mathcal{H}_k^\gamma), D\mathcal{H}_k^\gamma \rangle dX \right) = O(1) \quad (6.22)$$

as $k \rightarrow \infty$. Furthermore, reasoning as in the proof of Theorem 4.4, we show that the family of functions $\{\mathcal{U}_k^\gamma\}_{k \geq 1}$ is uniformly continuous in B_1 . Now we argue as follows, fix a $\gamma^* < 1$. From the uniform continuity, up to a subsequence,

$$\mathcal{U}_k^{\gamma^*} \rightarrow \mathcal{U} \quad \text{and} \quad \mathcal{H}_k^{\gamma^*} \rightarrow \mathcal{H}, \quad (6.23)$$

uniformly in $\overline{B_1}$. Also, $Y_k \rightarrow Y_0$. From (6.19) and (6.18), we obtain that

$$\begin{cases} \operatorname{div}(\mathcal{A}(Y_0, D\mathcal{H})) &= 0 & \text{in } B_1 \\ \mathcal{H} &= \mathcal{U} & \text{on } \partial B_1 \end{cases} \quad (6.24)$$

and that \mathcal{H} is the unique minimizer of

$$\mathcal{D}_0(v) := \int_{B_1} \langle \mathcal{A}(Y_0, Dv), Dv \rangle dX, \quad (6.25)$$

among functions $v \in W_0^{1,p}(B_1) + \mathcal{H}$. However, from (6.22), we obtain that

$$\mathcal{D}_0(\mathcal{U}) = \mathcal{D}_0(\mathcal{H}). \quad (6.26)$$

Therefore, $\mathcal{U} \equiv \mathcal{H}$. In particular, \mathcal{U} solves the elliptic PDE

$$\operatorname{div}(\mathcal{A}(Y_0, D\mathcal{U})) = 0 \text{ in } B_1.$$

However, since $\mathcal{U}(0) = 0$, by the strong maximum principle, $\mathcal{U} \equiv 0$, which ultimately contradicts (6.17) and the Theorem is finally proven. \square

Our next step is to prove that u_λ^* grows linearly away from $\partial\Omega_\lambda^*$. Notice that this is the most admissible growth rate allowed by the Lipschitz regularity previously proven in Theorem 6.1. Here is the precise statement:

Theorem 6.2. *There exists a constant $\underline{c} > 0$, depending on dimension \mathcal{A} , D , Γ and φ , such that*

$$\lambda^{-1/p} \underline{c} \cdot \operatorname{dist}(X_0, \partial\Omega_\lambda^*) \leq u_\lambda^*(X_0),$$

for any $X_0 \in \Omega_\lambda^*$.

Proof. Let us fix $X_0 \in \Omega_\lambda^*$ near the free boundary and label $d := \operatorname{dist}(X_0, \partial\Omega_\lambda^*)$. From Theorem 4.3, there exists a unique solution, ϕ , to the following obstacle problem

$$\operatorname{Min} \left\{ \int_{D^C} \langle \mathcal{A}(X, Df), Df \rangle dX \mid f \in W^{1,p}(D^C) \text{ } f = \varphi \text{ on } \partial D \text{ and } f \leq 0 \text{ in } \{u_\lambda^* = 0\} \cup B_{\frac{d}{2}}(X_0) \right\}. \quad (6.27)$$

Recall, in Theorem 4.4, we proved that u_λ^* is too a minimizer for problem $(\mathfrak{P}_\lambda^{\text{weak}})$, that is problem (2.13) and clearly ϕ competes with u_λ^* in such a problem; therefore

$$\int_{\partial D} (\Gamma(X, \partial_{\mathcal{A}}\phi) - \Gamma(X, \partial_{\mathcal{A}}u_\lambda^*)) d\mathcal{H}^{n-1}(X) \geq \lambda^{-1} c_n d^n, \quad (6.28)$$

for a dimensional constant c_n . Since both u_λ^* and ϕ are \mathcal{A} -harmonic in D_γ , where γ is the number in Proposition 4.2, for a constant $C_1 = C_1(\Gamma)$, we can estimate

$$\begin{aligned} \int_{\partial D} (\Gamma(X, \partial_{\mathcal{A}}\phi) - \Gamma(X, \partial_{\mathcal{A}}u_\lambda^*)) d\mathcal{H}^{n-1} &\leq C_1 \int_{\partial D} (\partial_{\mathcal{A}}\phi - \partial_{\mathcal{A}}u_\lambda^*) d\mathcal{H}^{n-1} \\ &\leq \frac{C_1}{\inf_{\partial D} \varphi} \int (\langle \mathcal{A}(X, D\phi), D\phi \rangle - \langle \mathcal{A}(X, Du_\lambda^*), Du_\lambda^* \rangle) dX. \end{aligned} \quad (6.29)$$

Here we have used the measure representation provided by Lemma 3.1. Now let h satisfy

$$\mathbb{L}h = 0 \text{ in } B_{\frac{2}{3}d}(X_0) \setminus B_{\frac{d}{2}}(X_0), \quad h = 0 \text{ in } B_{\frac{d}{2}}(X_0), \quad \text{and} \quad h = 1 \text{ on } \partial B_{\frac{2}{3}d}(X_0).$$

By the Harnack inequality, there exists a constant $c_2 > 0$, such that

$$u_\lambda^*(X) \geq c_2 u_\lambda^*(X_0) h(X) \text{ in } B_{\frac{2}{3}d}(X_0). \quad (6.30)$$

Consider the auxiliary function

$$\mathfrak{g}(X) := \begin{cases} \min \{u_\lambda^*(X), c_2 u_\lambda^*(X_0) h(X)\} & \text{in } B_{\frac{2}{3}d}(X_0) \\ u_\lambda^*(X) & \text{in } D^C \setminus B_{\frac{2}{3}d}(X_0). \end{cases}$$

Notice that \mathfrak{g} competes with ϕ in the obstacle problem, thus, combining (6.28), (6.29) and replacing ϕ by \mathfrak{g} , we obtain

$$\lambda^{-1} c_3 \leq \frac{1}{d^n} \int_{\Pi} (\langle \mathcal{A}(X, D\mathfrak{g}), D\mathfrak{g} \rangle - \langle \mathcal{A}(X, Du_\lambda^*), Du_\lambda^* \rangle) dX. \quad (6.31)$$

Where set set of integration in the above estimate can be taken to be

$$\Pi := \left\{ X \in B_{\frac{2}{3}d}(X_0) \setminus B_{\frac{1}{2}d}(X_0) \mid c_2 u_\lambda^*(X_0) h(X) \leq u_\lambda^*(X) \right\}.$$

However, in this set, we can estimate

$$\begin{aligned} \langle \mathcal{A}(X, D\mathbf{g}), D\mathbf{g}(X) \rangle &\leq \Lambda |D\mathbf{g}|^p \\ &\leq |Dh(X)|^p \cdot [c_2 u_\lambda^*(X_0)]^p \\ &\leq C \left[\frac{c_2 u_\lambda^*(X_0)}{d} \right]^p. \end{aligned} \tag{6.32}$$

In the last inequality we have used the $C^{1,\alpha}$ estimate for h . Finally, a combination of (6.31) and (6.32) leads us to

$$\lambda^{-1/p} \underline{c} d \leq u_\lambda^*(X_0),$$

for a constant $\underline{c} = \underline{c}(n, \mathcal{A}, D, \Gamma, \varphi)$, and the Theorem is proven. \square

Sometimes it is convenient to express nondegeneracy in any ball centered at a point $X_0 \in \overline{\Omega}_\lambda^*$. This is the contents of the next Theorem.

Theorem 6.3. *Let K be a compact set and $X_0 \in \overline{\Omega}_\lambda^* \cap K$. Then,*

$$\sup_{B_r(X)} u_\lambda^* \geq cr,$$

for some constant $c > 0$ depending on dimension, K , \mathcal{A} , D , Γ , φ and λ .

Proof. The proof is basically the same of as the proof of Theorem 6.2. The only difference is that u_λ^* is no longer \mathcal{A} -harmonic near a free boundary point X_0 , thus we replace the employment of Harnack inequality in (6.30) by:

$$v(X) := \sup_{B_r} u_\lambda^* \cdot h(X) \geq u_\lambda^* \text{ on } \partial B_r,$$

where h is the \mathcal{A} -harmonic function in $B_r \setminus B_{r/2}$ taking boundary data 1 on ∂B_r and 0 in $B_{r/2}$. We then define the auxiliary function $\mathbf{g}(X) := \min \{u_\lambda^*(X), v(X)\}$. The proof now follows the same path as in the proof of Theorem 6.2. \square

As usual, optimal regularity, Theorem 6.1 and nondegeneracy, Theorem 6.2 or Theorem 6.3, as you like, allow a deeper understanding on the geometric-measure properties of the free boundary. In the next Theorem we will show that the free boundary Ω_λ^* has the appropriate weak geometry.

Theorem 6.4. *There exists a constant $0 < \varsigma < 1$, depending on dimension, \mathcal{A} , D , Γ , φ , and $\lambda^{1/p}$, such that,*

$$\varsigma \omega_n r^n \leq \mathcal{L}^n(B_r(Z) \cap \Omega_\lambda^*) \leq (1 - \varsigma) \omega_n r^n, \tag{6.33}$$

for any ball $B_r(Z)$ centered at a free boundary point $Z \in \partial \Omega_\lambda^*$. Furthermore, the optimal configuration Ω_λ^* is a set of locally finite perimeter and for positive constants \underline{c} , \overline{C} , depending on \mathcal{A} , D , Γ , φ , and $\lambda^{1/p}$, there holds

$$\underline{c} r^{n-1} \leq \mathcal{H}^{n-1}(\partial \Omega_\lambda^* \cap B_r(Z)) \leq \overline{C} r^{n-1} \tag{6.34}$$

for any ball $B_r(Z)$ centered at a free boundary point. In particular, $\mathcal{H}^{n-1}(\partial \Omega_\lambda^* \setminus \partial_{\text{red}} \Omega_\lambda^*) = 0$.

Proof. The estimate by below in (6.33), that is, $\varsigma \omega_n r^n \leq \mathcal{L}^n(B_r(Z) \cap \Omega_\lambda^*)$, is an immediate consequence of Lipschitz regularity and strong nondegeneracy.

Let us focus our effort to prove the uniform density of the zero phase, $\mathbb{R}^n \setminus \Omega_\lambda^\star$. Let us assume, for purpose of contradiction, the existence of a sequence of positive real numbers r_j with $r_j \searrow 0$ as $j \rightarrow \infty$ and

$$\frac{\mathcal{L}^n(B_{r_j}(Z) \cap \{u_\lambda^\star = 0\})}{r_j^n} = O(1). \quad (6.35)$$

We consider then the blow-up sequence $q_j: B_1 \rightarrow \mathbb{R}$, defined as

$$q_j(Y) := \frac{1}{r_j} u_\lambda^\star(Z + r_j Y). \quad (6.36)$$

Let h_j be the solution to

$$\begin{cases} \operatorname{div}(\mathcal{A}(Z + r_j X, D h_j)) &= 0 \text{ in } B_1 \\ h_j &= q_j \text{ on } \partial B_1. \end{cases} \quad (6.37)$$

A renormalization of (4.13), when $p \geq 2$ or (4.14) when $1 < p \leq 2$, under the assumption (6.35), reveals

$$\int_{B_1} |\nabla(h_j - q_j)(Y)|^p dY = O(1). \quad (6.38)$$

By Lipschitz regularity of u_λ^\star , and $C^{1,\alpha}$ elliptic estimate, up to a subsequence, we may assume

$$q_j \xrightarrow{j \rightarrow \infty} q_0 \quad \text{and} \quad h_j \xrightarrow{j \rightarrow \infty} h_0. \quad (6.39)$$

uniformly in $B_{9/11}$. From (5.3) h_0 satisfies $\operatorname{div}(\mathcal{A}(Z, D h_0(Y))) = 0$, and from (6.38) so does q_0 , that is,

$$\operatorname{div}(\mathcal{A}(Z, D q_0(Y))) = 0 \text{ in } B_{1/2}. \quad (6.40)$$

Since $q(0) = 0$, by the strong maximum principle, we conclude $q(0) \equiv 0$ in $B_{1/2}$. However, this is a contraction on the nondegeneracy property guaranteed by Theorem 6.3.

We now turn our attention to (6.34). The estimate by above, that is $\mathcal{H}^{n-1}(\partial \Omega_\lambda^\star \cap B_r(Z)) \leq \overline{C} r^{n-1}$ is a consequence of Lipschitz regularity of u_λ^\star . In order to prove the estimate by below in (6.34), as before, let us assume, for the sake of contradiction, that there exists a sequence $r_j \searrow 0$ such that

$$\frac{\mathcal{H}^{n-1}(\partial \Omega_\lambda^\star \cap B_{r_j}(Z))}{r_j^{n-1}} = O(1). \quad (6.41)$$

With the notation as in (6.36), let us define the sequence of nonnegative measures ν_j , in $B_{2/3}$, as

$$\nu_j := \operatorname{div}(\mathcal{A}(Z + r_j X, D q_j)) dX. \quad (6.42)$$

Via a compactness argument, we may assume, modulo passing to a subsequence if necessary, that $\nu_j \rightharpoonup \nu_0$ in the sense of measures. However, condition (6.41) translates in terms of the measures ν_j as

$$\nu_j \rightharpoonup 0. \quad (6.43)$$

Moreover, by Lipschitz regularity, nondegeneracy and uniform positive density of both phases, estimate (6.33), it is not hard to verify that

$$\nu_j \rightharpoonup \nu_0 := \operatorname{div}(\mathcal{A}(Z, D q_0)) dX. \quad (6.44)$$

Indeed, from (6.33), $\mathcal{L}^n(\partial\{q_0 > 0\}) = 0$, thus in order to justify (6.44), it is enough to attest such an identity holds true for balls entirely contained in $\{q_0 > 0\}$ and in $\{q_0 = 0\}$. If $B \subset \{q_0 > 0\}$, then by elliptic estimate, q_j converges to q_0 in a $C^{1,\alpha}$ fashion in B . Thus clearly (6.44) is true. Now, if $B \subset \{q_0 = 0\}$, then

$$\left[\operatorname{div}(\mathcal{A}(Z, D q_0)) dX \right](B) = 0,$$

so we have to show that $\nu_j(B) \rightarrow 0$ as $j \rightarrow \infty$. This is a consequence of nondegeneracy. In fact, let $\tilde{B} \subset\subset B$. If there were a subsequence, q_{j_k} , for each $q_{j_k} \neq 0$ in \tilde{B} , then by Theorem 6.3, there should exist points $P_{k_j} \in \tilde{B}$, such that $q_{j_k}(P_{k_j}) \geq c > 0$. Then, passing to another subsequence, $P_{k_j} \rightarrow \bar{P} \in \tilde{B}$, and since q_{j_k} converges uniformly to q_0 , we would reach the conclusion that $q_0(\bar{P}) > c$, which is not possible. In conclusion, if B_k is a nested sequence of balls, with $B_k \nearrow B$, then, for some $j_k \in \mathbb{N}$, $q_j \equiv 0$ in B_k , for any $j > j_k$. Therefore, $\nu_j(B) \xrightarrow{j \rightarrow \infty} 0$, as desired.

Having verified (6.44), the observation in (6.43) tells us that

$$\operatorname{div}(\mathcal{A}(Z, Dq_0)) = 0 \text{ in } B_{2/3},$$

and as argued before, this leads us to a contradiction on the nondegeneracy feature of q_0 assured in Theorem 6.3. \square

An immediate, yet quite important consequence of Theorem 6.4 is a substantial enhancement of Lemma 3.1 for the measure $\mathbb{L}u_\lambda^*$.

Theorem 6.5. *There exists a Borel function Q_λ , such that $\mathbb{L}u_\lambda^* = Q_\lambda \lfloor \partial\Omega_\lambda^*$. That is,*

$$\int \operatorname{div}(\mathcal{A}(X, Du_\lambda^*)) \phi(X) dX = \int_{\partial\Omega_\lambda^*} Q_\lambda(S) \phi(S) d\mathcal{H}^{n-1}(S),$$

for any $\phi \in C_0^1(\mathbb{R}^n \setminus D)$. Moreover, Q_λ bounded away from zero and infinity, that is for a positive constant $C = C(\lambda, n, \mathcal{A}, D, \Gamma, \varphi)$, there holds

$$0 < C^{-1} \leq Q_\lambda \leq C < \infty.$$

As to provide some further insight, allow us to make some loose comments regarding the representation Theorem 6.5. The Borel function Q_λ should be understood as a weak notion for the $\partial_{\mathcal{A}}u_\lambda^*$ along the reduced free boundary $\partial_{\text{red}}\Omega_\lambda^*$. Indeed, in any C^1 peace of $\partial\Omega_\lambda^*$, there holds

$$Q_\lambda(S) = \langle \mathcal{A}(S, Du_\lambda^*(S)), \nu(S) \rangle, \quad (6.45)$$

where ν is the unit inward normal vector to $\partial\Omega_\lambda^*$ at S . However, $\nu(S) = \frac{\nabla u_\lambda^*(S)}{|\nabla u_\lambda^*(S)|}$, thus, taking into account the scaling feature of \mathcal{A} , property (c)(iv), from identity (6.45) we reach that

$$|\nabla u_\lambda^*(S)| = \sqrt[p-1]{\frac{Q_\lambda(S)}{\langle \mathcal{A}(S, \nu(S)), \nu(S) \rangle}}. \quad (6.46)$$

In a more rigorous way, expression (6.46) can be proven to hold in terms of an asymptotic approximation, that is, the following is true:

Theorem 6.6. *Let $X_0 \in \partial_{\text{red}}\Omega_\lambda^*$. Then, for any $X \in \Omega_\lambda^*$ near X_0 , we have*

$$u_\lambda^*(X) = \theta_\lambda(X_0) \langle X - X_0, \nu(X_0) \rangle^+ + o(|X - X_0|),$$

where $\theta_\lambda(X_0) = \sqrt[p-1]{\frac{Q_\lambda(X_0)}{\langle \mathcal{A}(X_0, \nu(X_0)), \nu(X_0) \rangle}}$.

Proof. Indeed, consider a convergent blow-up sequence

$$q_r(Y) := \frac{1}{r} u_\lambda^*(X_0 + rY) \xrightarrow{r \searrow 0} q_0. \quad (6.47)$$

Easily, from standard geometric-measures arguments, combined with nondegeneracy and the convergence in (6.47), we see that

$$q_0 \equiv 0 \text{ in } \{X \in \mathbb{R}^n \mid \langle X, \nu(X_0) \rangle < 0\} \quad \text{and} \quad \{q_0 > 0\} = \{X \in \mathbb{R}^n \mid \langle X, \nu(X_0) \rangle > 0\}. \quad (6.48)$$

Moreover

$$\operatorname{div}(\mathcal{A}(X_0, Dq_0), Dq_0) = 0 \text{ in } \{q_0 > 0\}. \quad (6.49)$$

Notice that $\partial\{q_0 > 0\}$ is the hyperplane $\{X \in \mathbb{R}^n \mid \langle X, \nu(X_0) \rangle = 0\}$: a smooth surface. One verifies from Theorem 6.5 that

$$\operatorname{div}(\mathcal{A}(X_0, Dq_0), Dq_0) = Q_\lambda(X_0) \llcorner \{X \in \mathbb{R}^n \mid \langle X, \nu(X_0) \rangle = 0\}, \quad (6.50)$$

hence, reasoning as before, we reach the following conclusion

$$\nabla q_0(Y) \cdot \nu(X_0) = \theta_\lambda(X_0), \quad \forall Y \in \{\langle X, \nu(X_0) \rangle = 0\}. \quad (6.51)$$

Recall q_0 is Lipschitz continuous in the entire \mathbb{R}^n . Let q_0^* be the odd reflection of q_0 with respect to the hyperplane $\{X \in \mathbb{R}^n \mid \langle X, \nu(X_0) \rangle = 0\}$. It is easy to verify that $\|\nabla q_0^*\|_{L^\infty(\mathbb{R}^n)} = \|\nabla q_0\|_{L^\infty(\mathbb{R}^n)} < C$ and that $\operatorname{div}(\mathcal{A}(X_0, Dq_0^*), Dq_0^*) = 0$ in the whole \mathbb{R}^n . From the $C^{1,\alpha}$ regularity of q_0^* , we can employ the beautiful and recent blow-up argument from [KSZ] to conclude that q_0^* is an affine function. Thus in view of (6.51), we obtain

$$q_0(X) = \theta_\lambda(X_0) \langle X - X_0, \nu(X_0) \rangle^+,$$

and the Theorem is proven. \square

We finish this section by proving the reduced free boundary, $\partial_{\text{red}}\Omega_\lambda^*$, admits a nice “stratification”. More precisely, we have

Theorem 6.7. *There exists a collection of C^1 hypersurfaces $\{\mathfrak{S}_j\}_{j \geq 1}$, and compact subsets $K_j \subset \mathfrak{S}_j$, such that*

$$\mathcal{H}^{n-1} \left(\partial_{\text{red}}\Omega_\lambda^* \setminus \bigcup_{j \geq 1} K_j \right) = 0.$$

Furthermore, if $X \in K_j$, the unit outward theoretical normal vector $-\nu(X)$ to $\partial_{\text{red}}\Omega_\lambda^*$ is normal to \mathfrak{S}_j .

Proof. Let $B = B_r(X_0)$ be a generically ball centered at a point of the reduced free boundary. By the Lipschitz continuity of u_λ^* and the ellipticity of \mathcal{A} , we know there exists a constant L , such that

$$\sup_B \mathcal{A}(X, Du_\lambda^*) \leq \frac{L}{6}. \quad (6.52)$$

Let \mathcal{I} be your favorite nonnegative radially symmetric smooth function whose support is B_1 . Normalize it so that $0 \leq \mathcal{I} \leq 1$; $\|\mathcal{I}\|_{L^1(B_1)} = 1$. Let \mathcal{I}_ϵ be the family of mollification induced by \mathcal{I} , that is, $\mathcal{I}_\epsilon(X) = \epsilon^{-n} \mathcal{I}(\epsilon^{-1}X)$. Also, select your favorite nonnegative function $\eta \in C_0^\infty(B)$, satisfying $\sup \eta = L^{-1}$. For sake of notation convenience, let us call $V(X) := \mathcal{A}(X, Du_\lambda^*)$. If ν denotes the Radon measure $D\chi_{\Omega_\lambda^*}$, we have, for $\epsilon \ll 1$,

$$\begin{aligned} \nu(B) &:= \sup \left\{ \int_{\Omega_\lambda^*} \operatorname{div} \psi dX \mid \psi \in C_0^1(B; \mathbb{R}^n), \|\psi\| \leq 1 \right\} \\ &\geq \int_{\Omega_\lambda^*} \operatorname{div} ((\eta V) * \mathcal{I}_\epsilon) dX \\ &= \int_{\Omega_\lambda^*} \mathcal{I}_\epsilon * \operatorname{div} (\eta V) dX \\ &= \int_{\Omega_\lambda^*} \operatorname{div} (\eta V) dX + O(1) \\ &= \int_{\Omega_\lambda^*} V \cdot \nabla \eta dX + \int_{\partial_{\text{red}}\Omega_\lambda^*} Q_\lambda(S) \eta(S) d\mathcal{H}^{n-1}(S) + O(1). \end{aligned} \quad (6.53)$$

Letting $\epsilon \rightarrow 0$ in (6.54) and afterwards letting $\eta \rightarrow L^{-1}$, we conclude there exists a constant $c(\lambda, \mathcal{A}, n, \Gamma, \varphi)$, such that

$$\nu(B) \geq c\mathcal{H}^{n-1}(B \cap \partial_{\text{red}}\Omega_\lambda^*). \quad (6.54)$$

In particular $\mathcal{H}^{n-1}|_{\partial_{\text{red}}\Omega_\lambda^*}$ is absolutely continuous with respect to $D\chi_{\Omega_\lambda^*}$. Now, arguing as in [DeGiorgi55] (see also [Giusti84] page 54 or [EG92] page 205) we prove the Theorem. \square

7 Existence of an optimal configuration for problem (2.4) in any dimension

In section 5, upon a restriction on the dimension, we have shown problem (2.4) has a minimal configuration. The strategy there was to let the penalizing parameter λ go to infinity and use appropriate estimates that becomes available under the constraint $n < p$, due to the Sobolev Imbedding Theorem.

The goal of this section is to explore the geometric-measure properties of the free boundary $\partial\Omega_\lambda^*$, established in the previous section, to settle to existence of an optimal design for problem (2.4) in all dimensions. However, as the readers should expect, the analysis here is rather more delicate as we will not be able to pass the limit on the penalty parameter λ . Instead, we will show that if we adjust the penalty term ϱ_λ properly, any optimal configuration, $\Omega^* = \Omega_\lambda^*$, for problem (2.10) will obey

$$\mathcal{L}^n(\Omega^* \setminus D) \leq \iota.$$

Therefore, Ω^* itself will be an optimal design for our primary optimization problem (2.4) and all the regularity features proven to hold for a solution to problem (2.10) will automatically extend to a solution to problem (2.4).

Before continuing, let us explain our strategy in a bit more technical terms. We will perform a small perturbation on an optimal configuration Ω_λ^* , around a point on the reduced free boundary: the portion of $\partial\Omega_\lambda^*$ where we can replace classical differential geometry arguments by geometric-measures ones. We will not compute the Borel function Q_λ of Theorem 6.5, as it is an extraordinary hard task: the free boundary condition for problem (2.10) is expected to be highly nonlocal. Instead, we will show that assuming $\mathcal{L}^n(\Omega_\lambda^* \setminus D) > \iota$ enforces a universal bound to the penalty parameter λ .

With the strategy well understood, let us establish the first supporting result towards the main goal of this section.

Lemma 7.1. *There exists a constant $M > 0$, depending on dimension, D , φ , Γ and \mathcal{A} , but independent of λ , such that*

$$\inf_{\partial_{\text{red}}\Omega_\lambda^*} Q_\lambda < M,$$

where Q_λ is the Borel function in Theorem 6.5.

Proof. Indeed, in the lights of Lemma 5.1, there exists a constant C , independent of λ , such that $\|u_\lambda^*\|_{W^{1,p}} \leq C$. Thus, from the Trace Theorem for Sobolev functions, we can write

$$\|u_\lambda^*\|_{W^{1,p}} \cdot [\mathcal{L}^n(\{u_\lambda^* > 0\})]^{\frac{1}{p'}} \geq \int_{\partial D} \varphi(Z) d\mathcal{H}^{n-1}(Z).$$

The above estimate combined with the Isoperimetric Inequality assures the existence of a constant $\underline{c}_1 > 0$, independent of λ , for which the following estimate holds

$$\mathcal{H}^{n-1}(\partial_{\text{red}}\Omega_\lambda^*) \geq \underline{c}_1. \quad (7.1)$$

From (7.1) and the representation in Theorem 6.5, we have

$$\begin{aligned} \int_{\partial D} \partial_{\mathcal{A}} u_{\lambda}^*(X) d\mathcal{H}^{n-1}(X) &= \int_{\partial_{\text{red}} \Omega_{\lambda}^*} Q_{\lambda}(X) d\mathcal{H}^{n-1}(X) \\ &\geq \underline{c}_1 \inf_{\partial_{\text{red}} \Omega_{\lambda}^*} Q_{\lambda}. \end{aligned} \quad (7.2)$$

Now, in view of estimate (5.4), for each $Y \in \partial D$ fixed, we establish the following estimate

$$\int_{\partial D} \Gamma \left(Y, \underline{c}_2 \cdot \left[\inf_{\partial_{\text{red}} \Omega_{\lambda}^*} Q_{\lambda} \right] \right) d\mathcal{H}^{n-1}(X) \leq \int_{\partial D} \Gamma \left(Y, \int_{\partial D} \partial_{\mathcal{A}} u_{\lambda}^* \right) d\mathcal{H}^{n-1}(X) \leq C_2.$$

Integrating the above estimate with respect to Y over ∂D and arguing as before, we conclude the proof of the Lemma. \square

We now pass to describe the mathematical setup for the suitable perturbation technique we shall employ on Ω_{λ}^* near a point on the reduced free boundary. Initially, select and fix, throughout this section, a free boundary point $Z_0 \in \partial_{\text{red}} \Omega_{\lambda}^*$, such that

$$Q_{\lambda}(Z_0) \leq 5 \inf_{\partial_{\text{red}} \Omega_{\lambda}^*} Q_{\lambda} \leq M_1, \quad (7.3)$$

where M_1 depends only on dimension, \mathcal{A} , D , Γ and φ , but it is independent of λ . The existence of such a point is guaranteed by Lemma 7.1.

Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be your favorite nonnegative smooth function whose support equals $[0, 1]$. Normalize it so that

$$\int \psi(\tau) d\tau = 1.$$

For a fixed positive, but small, real number α , we define the inward perturbation map around Z_0 as

$$\Phi_r(X) := \begin{cases} X - \alpha r \psi \left(\frac{|X - Z_0|}{r} \right) \nu(Z_0) & X \in B_r(Z_0) \\ X & X \notin B_r(Z_0). \end{cases} \quad (7.4)$$

Here, $\nu(Z_0)$ denotes the theoretical measure outward normal vector at Z_0 . The idea now is to compare Ω_{λ}^* with its inward perturbed configuration given by:

$$\Omega_r := \Phi_r(\Omega_{\lambda}^*). \quad (7.5)$$

For that, let us call u_r the \mathcal{A} -potential associated to Ω_r , that is, u_r is the solution to

$$\begin{cases} \mathcal{L} u_r &= 0 \text{ in } \Omega_r \setminus D \\ u_r &= \varphi \text{ on } \partial D \\ u_r &= 0 \text{ on } \partial \Omega_r \end{cases} \quad (7.6)$$

Although it is possible to compare u_r and u directly, it turns out to be more convenient to use the auxiliary function, v_r , implicitly by

$$v_r(\Phi_r(X)) = u_{\lambda}^*(X). \quad (7.7)$$

Notice that $(\{v_r > 0\}, v_r)$ is not suitable for our minimization problem (2.10). Also it is not efficient to compare it with u_{λ}^* in terms of the minimization problem (2.13), since $\partial_{\mathcal{A}} u_{\lambda}^* \equiv \partial_{\mathcal{A}} v_r$. Our strategy is to compare v_r with u_{λ}^* and with u_r separately and then combine these information using v_r as a bridge from u_r and u_{λ}^* .

The next two Lemmas are from [OT06], Section 4, though in that paper the computations are carried out only for the p -Laplacian operator. Thus we decide to include in this present work “economic versions” of their proofs as a courtesy to the readers.

Lemma 7.2. *With the notation previously set, we have*

$$\mathcal{L}^n(\{u > 0\}) - \mathcal{L}^n(\{v_r > 0\}) = M_2 \alpha r^n + o(r^n),$$

for a universal constant $M_2 > 0$.

Proof. For sake of notation convenience, we will write u for u_λ^* . For each $r > 0$ small, we consider the r -normalization of u around Z_0 , $u_r: B_1 \rightarrow \mathbb{R}$, defined as

$$u_r(Y) := \frac{1}{r} u(Z_0 + rY).$$

Since $Z_0 \in \partial_{\text{red}} \Omega_\lambda^*$,

$$B_1 \cap \{u_r > 0\} \xrightarrow{r \rightarrow 0} \{Y \in B_1 \mid \langle Y, \nu(Z_0) \rangle < 0\}, \quad (7.8)$$

in the sense that the characteristic functions of the above sets in the LHS converge to the characteristic function of the set in the RHS in the $L_{\text{loc}}^1(\mathbb{R}^n)$ topology. One easily sees, by the Change of Variables Theorem, that

$$\begin{aligned} \frac{\mathcal{L}^n(B_r(Z_0) \cap \{v_r > 0\})}{r^n} &= \frac{1}{r^n} \int_{B_r(Z_0) \cap \{v_r > 0\}} dX \\ &= \int_{B_1 \cap \{v_r(Z_0 + rY) > 0\}} dY \\ &= \int_{B_1 \cap \{u_r > 0\}} \det(D\Phi_r(Z_0 + rY)) dY \\ &\xrightarrow{r \rightarrow 0} \int_{B_1 \cap \{\langle Y, \nu(Z_0) \rangle < 0\}} 1 - \alpha \psi'(|Y|) \left\langle \frac{Y}{|Y|}, \nu(Z_0) \right\rangle dY, \end{aligned} \quad (7.9)$$

It is important to highlight that for any unit vector $\nu \in \mathbb{S}^{n-1}$,

$$\int_{B_1 \cap \{\langle Y, \nu \rangle < 0\}} \psi'(|Y|) \left\langle \frac{Y}{|Y|}, \nu \right\rangle dY \equiv M_2, \quad (7.10)$$

where M_2 is a constant that depends only on your choice for ψ . Similarly, one finds that

$$\frac{\mathcal{L}^n(B_r(Z_0) \cap \{u_\lambda^* > 0\})}{r^n} \xrightarrow{r \rightarrow 0} \int_{B_1 \cap \{\langle Y, \nu(Z_0) \rangle < 0\}} dY. \quad (7.11)$$

Combining (7.8), (7.9), (7.10) and (7.11), we conclude the Lemma. \square

Our next Lemma measures the differential on the \mathcal{A} -Dirichlet integral passing from u_λ^* to v_r .

Lemma 7.3. *There exists a constant $M_3 > 0$ depends on dimension, D , Γ , φ and ψ , but it is independent of λ such that*

$$\frac{1}{r^n} \int \{\langle \mathcal{A}(X, Dv_r), Dv_r \rangle - \langle \mathcal{A}(X, Du_\lambda^*), Du_\lambda^* \rangle\} dX \leq \alpha M_3 + o(\alpha) + O(1).$$

Proof. Again, for sake of notation convenience, we will write u for u_λ^* . Yet for notation convenience, let us write, for any vector field \vec{V} , $\Theta(\vec{V})(X) := \langle \mathcal{A}(X, \vec{V}), \vec{V} \rangle$. Applying the Change of

Variables Theorem twice and taking into account that P_r maps $B_r(X_i)$ diffeomorphically onto itself, we can write

$$\begin{aligned}
\frac{1}{r^n} \int_{B_r(Z_0)} \Theta(Dv_r)(X) dX &= \frac{1}{r^n} \int_{B_r(Z_0)} \Theta(D\Phi_r(\Phi_r^{-1}(X))^{-1} \cdot \nabla u(\Phi_r^{-1}(X))) dX \\
&= \frac{1}{r^n} \int_{B_r(Z_0)} \Theta(D\Phi_r(Y)^{-1} \cdot \nabla u(Y)) \times |\det(D\Phi_r(Y))| dY \\
&= \int_{B_1 \cap \{u_r > 0\}} \Theta(D\Phi_r(Z_0 + rZ)^{-1} \cdot \nabla u_r(Z)) \times |\det(D\Phi_r(Z_0 + rZ))| dZ.
\end{aligned} \tag{7.12}$$

By an explicit computation it is easy to verify that

$$D\Phi_r(Z_0 + rZ)^{-1} \cdot \nabla u_r^i(Z) = \nabla u_r(Z) + \alpha \frac{\psi'(|Z|)}{|Z|} \langle Z, \nabla u_r(Z) \rangle \nu(Z_0) + o(\alpha). \tag{7.13}$$

Furthermore, we can compute explicitly that

$$|\det(D\Phi_r(Z_0 + rZ))| = 1 - \alpha \frac{\psi'(|Z|)}{|Z|} \langle Z, \nu(Z_0) \rangle. \tag{7.14}$$

A straight combination of (7.12), (7.13) and (7.14), reveals that

$$\frac{1}{r^n} \int_{B_r(Z_0)} \Theta(Dv_r)(X) - \Theta(Du)(X) dX = -\alpha \int_{B_1 \cap \{u_r > 0\}} \Theta(Du_r(Z)) \frac{\psi'(|Z|)}{|Z|} \langle Z, \nu(Z_0) \rangle dZ + o(\alpha). \tag{7.15}$$

It is simple to verify, from the Divergence Theorem, that

$$\int_{B_1 \cap \{u_r > 0\}} \frac{\psi'(|Z|)}{|Z|} \langle Z, \nu(Z_0) \rangle dZ \longrightarrow - \int_{B_1 \cap \{\langle Z, \nu(Z_0) \rangle = 0\}} \psi(|Z|) d\mathcal{H}^{n-1}(Z) = I > 0, \tag{7.16}$$

with the appropriate integral orientation. Furthermore, by the Lipschitz regularity of u and standard geometric-measure arguments we verify that

$$\langle \mathcal{A}(Z_0 + rZ, \nabla u_r), \nabla u_r \rangle \rightarrow Q_\lambda(Z_0) \nu(Z_0) \chi_{B_1 \cap \{\langle Y, \nu(X_i) \rangle < 0\}}, \tag{7.17}$$

in $L^p(B_1)$. Thus, letting $r \rightarrow 0$ in (7.15), and taking into account (7.16) and estimate (7.3), we conclude the proof of the Lemma. \square

We are ready to prove the existence of an optimal design for problem (2.4) in all dimensions.

Theorem 7.4. *There exists a positive number λ_0 , such that if Ω_λ^* is an optimal configuration for problem (2.10) and $\mathcal{L}^n(\Omega_\lambda^* \setminus D) > \iota$, then necessarily, $\lambda < \lambda_0$. In particular, there exists an optimal configuration for problem (2.4) and it enjoys all the weak geometric features derived in Section 6.*

Proof. Throughout the proof we fix an optimal configuration Ω_λ^* and assume

$$\mathcal{L}^n(\Omega_\lambda^* \setminus D) > \iota. \tag{7.18}$$

Initially we recall the variational characterization of the \mathcal{A} -potential u_r , namely

$$\int \langle \mathcal{A}(X, Du_r), Du_r \rangle dX = \min \left\{ \int \langle \mathcal{A}(X, Dv), Dv \rangle dX \mid v = \varphi \text{ on } \partial D \text{ and } v = 0 \text{ on } \partial\Omega_r \right\}. \tag{7.19}$$

Now we compare Ω_λ^* with Ω_r in terms of the minimization problem (2.10). From the minimality feature of the configuration Ω_λ^* , if r is small enough as to $\mathcal{L}^n(\Omega_r \setminus D) > \iota$, we have

$$\lambda \{ \mathcal{L}^n(\Omega_\lambda^* \setminus D) - \mathcal{L}^n(\Omega_r \setminus D) \} \leq \int_{\partial D} \Gamma(X, \partial_{\mathcal{A}} u_r) - \Gamma(X, \partial_{\mathcal{A}} u_\lambda^*) d\mathcal{H}^{n-1}(X). \quad (7.20)$$

As argued before, we have the following estimate

$$\begin{aligned} \int_{\partial D} \Gamma(X, \partial_{\mathcal{A}} u_r) - \Gamma(X, \partial_{\mathcal{A}} u_\lambda^*) d\mathcal{H}^{n-1}(X) &\leq C(\partial D, \Gamma) \int_{\partial D} \{ \partial_{\mathcal{A}} u_r - \partial_{\mathcal{A}} u_\lambda^* \} d\mathcal{H}^{n-1}(X) \\ &\leq C(\partial D, \Gamma, \inf \varphi) \int \langle \mathcal{A}(X, Du_r), Du_r \rangle \\ &\quad - \langle \mathcal{A}(X, Du_\lambda^*), Du_\lambda^* \rangle dX. \end{aligned} \quad (7.21)$$

Now combining Lemmas 7.2 and 7.3 with (7.19), (7.20) and (7.21), we obtain

$$\lambda \{ M_2 \alpha r^n + o(r^n) \} \leq C(\partial D, \Gamma, \inf \varphi) r^n \times [\alpha M_3 + o(\alpha) + O(1)]. \quad (7.22)$$

If we divide expression (7.22) by r^n , let $r \rightarrow 0$ and afterwards divide the result by α and let $\alpha \searrow 0$, we finally conclude the proof of the Theorem. \square

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